Assignment 3 Solutions

Q1: 20%

The A_{21} coefficient is defined as the reciprocal of the lifetime of the state

```
q1=Solve[tau==1/a21 /. tau->2.3 10^-9, a21]
```

$\{\{a21 \to 4.34783 \times 10^{8}\}\}$

To make this more clear we can directly integrate the equation delivering us the number density of n2 as a function of time where the rate of decay follows Einstein's prescription of being proportional to the product of the A21 coefficient and the number density in that state.

```
DSolve[{n2[0] == 1, D[n2[t], t] == A21 n2[t]}, n2[t], t]
```

$\{\{n2[t] \to e^{A21\,t}\}\}$

We see that when t = 1/A21 we have arrived at a residual population of 1/e of its starting amount - i.e. the lifetime of the level.

So first calculating the transition frequency corresponding to 10.22eV:

```
q3=Solve[  nu==(q V)/h /.
{q->1.6 10^-19, V->10.22, h->6.63 10^-34},nu]
```

$\{\{nu \to 2.46637 \times 10^{15}\}\}$

And then using the relationship between A and B to calculate B:

```
Solve[a21==b12 8 Pi h nu^3/c^3 /.
{h->6.63 10^-34,First[First[q3]],c->3 10^8,
First[First[q1]]},b12]
```

$\{\{b12 \to 4.69581 \times 10^{19}\}\}$

So the B_{12} Einstein coefficient is 4.7 x 10^{19} .

Despite my nasty question the value for B21 is just exactly the same as the value of B12. Of course the temperature and density should be irrelevant to an atomic constant

2 (b) 40%

Let's write the rate equations for the conventional three level scheme (2b):

```
dn3dt= wp(n1-n3) - a32 n3
```

```
-(a32 n3) + (n1 - n3) wp
```

```
dn2dt = wl (n1-n2) - a21 n2 + a32 n3
```

```
-(a21 n2) + a32 n3 + (n1 - n2) wl
```

```
dn1dt = wp (n3-n1) + a21 n2 + wl (n2-n1)
```

```
a21 n2 + (-n1 + n2) wl + (-n1 + n3) wp
```

Before solving the equations lets make them simpler by substituting for a zero laser intensity (imagine that the threshold pumping level is sufficient to create a population inversion but insufficient to produce significant radition density corresponding to the 2-> 1 transition)

Finding n_2 in terms of the population densities of the other levels:

```
soln1=Solve[dn1dt==0 /. wl->0,n2]
```

$$\{\{n2 \to -(\frac{(-n1 + n3)\ wp}{a21})\}\}$$

Finding n_3 in terms of the population density of level 1:

```
soln2=Solve[dn3dt==0,n3]
```

$$\{\{n3 \to -(\frac{n1\ wp}{-a32 - wp})\}\}$$

Finding n_2 in terms of the population density of level 1:

```
soln3=soln1 /. First[soln2]
```

$$\{\{n2 \to -(\frac{wp\ (-n1 - \frac{n1\ wp}{-a32 - wp})}{a21})\}\}$$

Write the population difference of levels 2 and 1 in terms of the population density of level 1.

```
Simplify[(n2-n1)/(n1+n2+n3) /. {
soln2 [[1]][[1]],
soln3 [[1]][[1]]}]
```

$$\frac{-(a21\ a32) - a21\ wp + a32\ wp}{a21\ a32 + 2\ a21\ wp + a32\ wp}$$

Making the usual assumption that $A_{32} >> A_{21}$ we can simplify:

```
% /. { a21 wp->0 }
```

$$\frac{-(a21\ a32) + a32\ wp}{a21\ a32 + a32\ wp}$$

```
Simplify[%]
```

$$\frac{-a21 + wp}{a21 + wp}$$

Thus the pumping rate must exceed the spontaneous decay rate of level 2 to 1 before we can have a population inversion.

2 (a)

Write down the rate equations which govern the unconventional first 3 level scheme (2a)

```
dn3dt=wp (n1-n3) + wl (n2-n3) - a32 n3
```

```
-(a32 n3) + (n2 - n3) wl + (n1 - n3) wp
```

```
dn2dt = wl (n3-n2) + a32 n3 - a21 n2
```

```
-(a21 n2) + a32 n3 + (-n2 + n3) wl
```

```
dn1dt = a21 n2 + wp (n3 -n1)
```

```
a21 n2 + (-n1 + n3) wp
```

Writing level 3 population density in terms of level 2 population density (once again let‘s assume that the laser intensity is zero at the threshold pump power)

```
Simplify[Solve[dn2dt == 0 /. wl->0 ,n3]]
```

$$\{\{n3 \to \frac{a21\ n2}{a32}\}\}$$

Writing level 1 population density in terms of the population density of the other two levels.

```
Simplify[Solve[dn1dt==0 ,n1]]
```

$$\{\{n1 \to n3 + \frac{a21\ n2}{wp}\}\}$$

Writing level 1 population density in terms of the population density of level 2.

```
Simplify[% /. {First[%%]}]
```

$$\{\{\{n1 \to \frac{a21\ n2}{a32} + \frac{a21\ n2}{wp}\}\}\}$$

Writing the population difference between levels 3 and 2 in terms of level 2 population

```
pd=Simplify[ (n3-n2)/(n1+n2+n3) /. {
n3 -> (n2*a21)/a32,
n1 -> (n2*a21)/a32 + (a21*n2)/wp}]
```

$$\frac{(a21 - a32)\ wp}{a21\ a32 + 2\ a21\ wp + a32\ wp}$$

But since the lifetime of the bottom state is much shorter than the upper state (A_{21}>>A_{32}) we can simplify to:

```
% /. {(a21 - a32)->a21,
a32 wp ->0}
```

$$\frac{a21\ wp}{a21\ a32 + 2\ a21\ wp}$$

What is the threshold pump power?

```
Simplify[%]
```

$$\frac{wp}{a32 + 2\ wp}$$

So we see that the pump threshold for the second case occurs at zero power. The physics of this case strongly resembles that of the conventional four level laser scheme since we only need to lift a single atom into an excited state to create a population inversion. The second 3 level scheme is in fact more efficient than that of the conventional scheme (in terms of need for pump power). Nevertheless, I am unaware of any laser which actually operates using this scheme. This is no doubt because, for a good laser, one requires that the two laser levels should be very narrow in energy terms (to deliver a monochromatic output), while in order to get efficient pump absorption it is preferable for the upper level of the pump transition to have some broadness (because the pump energy is only rarely monochomatic). As a second point, for a pair of levels to strongly couple to the pump radiation (and therefore extract its energy efficiently) it is necessary for the transition to have a large B value - and as we found in the answer to question 1 this implies that the A value for this transition is also large. However, in order to build a nice laser transition it is necessary for the upper level of the transition to be long lived (in order to accumulate atoms in that level for the population inversion). In scheme (b) (or in a four level laser system) we can achieve all these things simultaneously (i.e. broad, strongly coupled upper pump level and narrow, long-lived upper laser level) while in scheme (a) this is impossible because level three is both a pump and laser level, and level 2 must be short-lived to clear out the bottom state so that inversion can occur.

3(a) 5%

Cavity lifetime and photon lifetime both have the same numerical value. The first is thought of as the length of time till the energy in the cavity is down to 1/e of its former amount while the second is the average time until a photon escapes from the cavity. The cavity lifetime is given by (in the absence of any losses apart from the reflectivity of the mirrors):

```
q3a=Solve[tc==2 d/(-c Log[r1 r2]) /. {

d->.12, c->3 10^8, r1->.97, r2->.97},
tc]
```

$\{\{tc \to 1.31323 \times 10^{-8}\}\}$

For the values given in this problem the cavity lifetime is 13 ns.

3(b)(ii) 10%

The usual approx. is to assume that the spontaneous emission curve is like a top-hat function. Since g(ν) is normalised this means the value of the function at any point inside the gain curve is merely the reciprocal of the width of the gain curve. Outside the gain curve the value of the function is zero i.e.

```
q32=Solve[gnur== 1/ fwhm /. {
fwhm->1.25 10^9},
gnur] //N
```

$\{\{gnur \to 8. \times 10^{-10}\}\}$

3(c) 10%

Using equation 9.4 from the notes:

```
q94=Solve[deltan == (8 Pi nur^2 tspont)/(c^3 gnur tc) //.
{c->3 10^8, nur-> c/(633.8 10^-9) , tspont->190 10^-9,
First[First[q32]],First[First[q3a]]},deltan]
```

$\{\{deltan \to 3.77169 \times 10^{15}\}\}$

3(d) 10%

Threshold pump power is given by:

$W_{41}\, n_1\, h\, \nu_p$ (from equation on p. 126)

but at threshold almost all the atoms are in the ground state so this can be approximated by:

$W_{41}\, N\, h\, \nu_p$

To find W_{41} we use equation 10.24, however as only a very small fraction of atoms are in the excited states (4×10^{15} m^{-3} are in excited states from the answer to the last question while for gas densities of 1/1000 of an atmosphere and normal temperatures there would be about 10^{22} Ne atoms/m^3) we can approximate equation 10.24 as:

$$qdi = Solve\left(\frac{deltan}{n} == \frac{w41}{a32}, n\right)$$

$$\left\{\left\{n \to \frac{a32\ deltan}{w41}\right\}\right\}$$

because A_{32} must be >> W_{41}. Substituting this expression back into the pump power threshold expression we get:

```
ans=w41 n h nup /. First[First[qdi]]
```

```
a32 deltan h nup
```

So substituting in the known values we get:

```
ans /.
{a32->1/(210 10^-9) , First[First[q94]],
h->6.6 10^-34, nup->5 10^15}
```

```
59 269.5
```

So the pump power required for the laser is 59 kW m^{-3} or 59mW cm^{-3}

(3e) 5%

The level of pump power required is less by a factor of 10 than the level required for a Nd:YAG laser (an example of a continuous wave (CW) laser) and around 10^4 times less than that required for a typical three level laser. Since dissipation of pump power is the major limitation for a CW laser, a He:Ne laser would appear to be an excellent CW laser candidate.