Assignment 2 Solutions

MODERN OPTICS EXAMINATION HINTS

With regard to the Modern Optics examination please note:

- It is 1.5 hours long.There will be three questions of which it is necessary to do all three. Remember that the average assignment mark can replace the poorest mark given to one of these three questions (of course this will only happen if the average assignment mark exceeds the mark given for the worst performed question.)

- know how to perform simple mathematical operations i.e. Differentiation / Finding Eigenvalues etc as there will be no *Mathematica* available in the exam.

- The exam will test understanding rather than memory work. I will provide the most difficult and large formula on the front page of the exam.

- Remember to apply approximations at the earliest opportunity as they will significantly reduce the time necessary to complete the question.

- Make sure you know how to do all the assignment questions, the past exam questions, and the derivations in the notes. If you don‘t know how to do them then come and see me (FSM Lab in the basement: Phone # 7028)

- The assignments have only tested to the end of section 10 of the notes. Please make sure that you have looked beyond this point in your study.

- and finally, Good Luck!

1(a) 5% The g factors of the resonator are:

$$g_{\text{one}} = \text{Solve}\left(g_1 == 1 - \frac{l}{r_1} \;/.\; \{l \to 1,\, r_1 \to .93 * 2\}\right)$$

```
{{g1 → 0.462366}}
```

$$g_{\text{two}} = \text{Solve}\left(g_2 == 1 - \frac{l}{r_2} \;/.\; \{l \to 1,\, r_2 \to 1.5 * 2\}\right)$$

```
{{g2 → 0.666667}}
```

The product of these g factors is:

$$(g_1 \;/.\; \text{First}[g_{\text{one}}])\,(g_2 \;/.\; \text{First}[g_{\text{two}}])$$

```
0.308244
```

Since this falls between 0 and 1 the resonator is stable

1(b) 15% Mirror 1 position - (waist is defined to be at z=0)

$$\mathbf{waistpossy = z1 == \frac{r2 - \mathit{l}^2}{2\,\mathit{l} - r1 - r2}}$$

$$z1 == \frac{-l^2 + r2}{2\ l - r1 - r2}$$

$$\mathbf{(z1\ /.\ Solve[waistpossy, z1])\ /.\ \{r1 \to .93 \times 2,\ r2 \to 1.5 \times 2,\ \mathit{l} \to 1\}\ \ //\ First}$$

```
-0.699301
```

So the waist falls at 70 cm from the more highly curved mirror and only 30 cm from the more weakly curved mirror. This seems counter-intuitive (indeed, I thought it would be the other way round) but a bit of thought shows that it must be the case. Generally, for most stable resonators the length of the resonator is approximately equal to the Rayleigh range. Therefore the Gaussian beam **cannot** be thought of as a spherical beam. For distances closer to the waist than the Rayleigh range, the beam curvature is larger close to the waist and decreases as we move away from the waist (the opposite of the behaviour found for distances further from the waist than the Rayleigh range) - the curvature is a minimum at the Rayleigh range. Therefore the least curved mirror must be closer to the waist so as to match the beam curvature! This is not always the case but is usually true for most resonators encountered in practice.

Rayleigh Range in terms of the resonator parameters

$$\mathbf{zr^2 == \frac{\mathit{l}\,(r1 - \mathit{l})\,(r2 - \mathit{l})\,(-\mathit{l} + r1 + r2)}{(-2\,\mathit{l} + r1 + r2)^2}}$$

$$zr^2 == \frac{l\ (-l + r1)\ (-l + r2)\ (-l + r1 + r2)}{(-2\ l + r1 + r2)^2}$$

```
Solve[% /. {r1→.93 2,r2→1.5 2,l->1},zr]
```

```
{{zr → -0.900932}, {zr → 0.900932}}
```

So the Rayleigh range is around 0.9 m. Outside this distance a Gaussian beam can usually be thought of as a very close approximation to a spherically expanding beam.

Finding waist of fundamental axial resonator mode corresponding to the Rayleigh range

$$\mathbf{Solve\left[zr == \frac{\pi\, w0^2}{lambda}\ /.\ \left\{lambda \to \frac{850}{10^9},\ zr \to 0.9\right\},\ w0\right]}$$

```
{{w0 → -0.000493464}, {w0 → 0.000493464}}
```

Waist of fundamental resonator axial mode is 493 μm in diameter (for a wavelength of 830nm this is 487 μm)

Finding spot size on more highly curved mirror

$$\mathbf{Solve}\left[\mathbf{w1}^2 == \mathbf{w0}^2\left(\left(\frac{\mathbf{z1}}{\mathbf{zr}}\right)^2 + 1\right) \mathbin{//.} \{\mathbf{w0} \to 493 \times 10 \,\hat{}\, {-6},\ \mathbf{z1} \to -0.699,\ \mathbf{zr} \to 0.9\},\ \mathbf{w1}\right]$$

```
{{w1 → -0.000624227}, {w1 → 0.000624227}}
```

So spot size is 624 μm on this mirror

Finding spot size on less highly curved mirror

$$\mathbf{Solve}\left[\mathbf{w2}^2 == \mathbf{w0}^2\left(\left(\frac{\mathbf{z1}}{\mathbf{zr}}\right)^2 + 1\right) \mathbin{//.} \{\mathbf{w0} \to 493 \times 10 \,\hat{}\, {-6},\ \mathbf{z1} \to 0.3,\ \mathbf{zr} \to 0.9\},\ \mathbf{w2}\right]$$

```
{{w2 → -0.000519668}, {w2 → 0.000519668}}
```

Spot size on this mirror is 520 μm

1(c) 15%
What is the corresponding Rayleigh Range for the laser beam waist size?

$$N\left[\mathbf{Solve}\left[\mathbf{zr} == \frac{\pi\,\mathbf{wl}^2}{\mathbf{lambda}} \mathbin{/.} \left\{\mathbf{lambda} \to \frac{830}{10^9},\ \mathbf{wl} \to \frac{120}{10^6}\right\},\ \mathbf{zr}\right]\right]$$

```
{{zr → 0.0545047}}
```

Rayleigh range for the laser beam is 5.4 cm

Where do the beam spot sizes match?
(wl is the laser waist,
zrl is the laser Rayleigh range,
w0 is the resonator waist,
zr0 is the resonator Rayleigh range,
z3 measures the distance from the laser waist)

N.B. I have altered the distance on the RHS of the expression to reflect the fact that the waists aren't at the same position. Remember that the distance used in these expressions means the distance from each respective waist. I made a mistake in the assignment and had the laser and cavity wavelength slightly different - whatever you have done about this I will mark it right as it was my mistake.

$$\mathbf{Solve}\left[\mathbf{wl}^2\left(\left(\frac{\mathbf{z3}}{\mathbf{zrl}}\right)^2+\mathbf{1}\right)==\mathbf{w0}^2\left(\left(\frac{-\mathbf{z3}+\frac{.88}{2}+\mathbf{3}+\mathbf{0.7}}{\mathbf{zr0}}\right)^2+\mathbf{1}\right)\ \text{/. }\left\{\mathbf{wl}\to\frac{\mathbf{120}}{\mathbf{10^6}},\ \mathbf{w0}\to\frac{\mathbf{487}}{\mathbf{10^6}},\ \mathbf{zrl}\to\mathbf{0.0545},\ \mathbf{zr0}\to\mathbf{0.9}\right\},\ \mathbf{z3}\right]$$

```
{{z3 → -1.37128}, {z3 → 0.839064}}
```

```
-.8177 + .88 / 2 + 3 + .7
```

```
3.3223
```

So the spot sizes match in two positions: One which is 0.84 metres from the laser waist in the direction of the resonator - thus being 3.32 metres from the resonator waist. The other point doesn‘t lie between the two resonators. However, it would still be possible to mode-match from this point using a curved mirror. I will ignore this solution from now on though.

If instead of doing the full glory Gaussian beam analysis, I had assumed we were sufficiently far from the waists that I could have used the approximations given to you in the notes then the answer would have come out as:

$$\mathbf{Solve}\left[\frac{\lambda\,\mathbf{z3}}{\pi\,\mathbf{wl}}==\left(\frac{\lambda\left(-\mathbf{z3}+\frac{.88}{2}+\mathbf{3}+\mathbf{0.7}\right)}{\pi\,\mathbf{w0}}\right)\ \text{/. }\left\{\mathbf{wl}\to\frac{\mathbf{120}}{\mathbf{10^6}},\ \mathbf{w0}\to\frac{\mathbf{487}}{\mathbf{10^6}},\ \mathbf{zrl}\to\mathbf{0.0545},\ \mathbf{zr0}\to\mathbf{0.9}\right\},\ \mathbf{z3}\right]$$

```
{{z3 → 0.818451}}
```

So I guess one can see that this is a pretty good approximation in this case - and we will thus use this approach for the next section (use the correct distances but make use of the approximation) ... Experimentally it is very difficult to locate optical components any better than that anyway.

1(d) 15%

What is the (absolute value) of the radius of curvature for the two beams at the point where their spot sizes match?

- **Since we are finding the radius of curvature for the beam well outside the Rayleigh Range we can considerably simplify the problem as the radius of curvature at these points is just exactly the distance it has travelled. Thus:**

```
Solve(rlaser == z /. {z → 0.83906}, rlaser)
```

```
{{rlaser → 0.83906}}
```

```
Solve(rcavity == z /. {z → −.8177 + .88 / 2 + 3 + .7 }, rcavity)
```

```
{{rcavity → 3.3223}}
```

N.B. One of these beams is converging (the resonator mode) and one of these beams is diverging (the laser beam). Thus the laser beam curvature should be positive (the rays representing the spherical wavefront would have a positive ratio for their perpendicular distance over their slope), and the resonator mode curvature negative

We can represent a spherical wavefront using the ray matrix methods as a vector [x x/R] where R is the radius of curvature of the beam (assuming that the paraxial approximations hold true). What effect does a lens have on such a beam?:

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}.\left\{x, \frac{x}{R}\right\} \text{ // MatrixForm}$$

$$\begin{pmatrix} x \\ -\frac{x}{f} + \frac{x}{R} \end{pmatrix}$$

This represents a new spherical beam with a radius of curvature: (f R)/(f-R)

Solving the lens equation to find the appropriate lens to transform the curvatures in the correct fashion.

$$\text{Solve}\left(\text{rnew} == \frac{f\ \text{rold}}{f - \text{rold}} \text{ /. } \{\text{rnew} \to -3.3223, \text{rold} \to .83906\}, f\right)$$

```
{{f → 0.669879}}
```

So the lens should be a convex (focussing) lens with a focal length of 67 cm.

Q 2 20%

Twice the cavity length is the time between output pulses multiplied by the speed of light:

$$N\left(\text{Solve}\left(2\, l == c\, t \text{ /. } \left\{c \to 3\, 10^8 / 1.03,\ t \to \frac{5}{10^9}\right\}, l\right)\right)$$

```
{{l → 0.728155}}
```

Cavity length =0.73 cm

How many pulses does it take to decay to 1/e of its former intensity?

```
35 / 5 // N
```

```
7.
```

Around 7 pulses.

Imagine two identical mirrors with power reflectivity of R. Radiation making one round trip around such a cavity will be reduced in intensity by R^2 as well as a factor exp(-α L 2) for the two trips it has made through the lossy material for a full round trip. After n round trips the intensity will be (R^{2n}exp (-n α L 2)) times down of its initial intensity. However we also know that after 7 round trips the intensity was 1/e of its initial intensity. Thus:

```
(R^2 Exp[-.03 .73×2]) == (1 / E) ^ (1 / 7) // Solve
```

```
{{R → -0.951678}, {R → 0.951678}}
```

Thus the mirror power reflectivity is 95%. The amplitude reflectivity is the squre root of this number. If we want to figure out the next things then we need to be careful and make sure we use the original definitions of Q factor or finesse - otherwise we are going to make some mistakes.... The photon lifetime is just numerically the same thing as the time constant of the pulse decrease i.e. it is 35ns. So from this we can figure out the linewidth of one resonance...the angular frequency width (FWHM) of one of the cavity modes is exactly equal to the reciprocal of the photon lifetime so we have...FWHM = 4.5 MHz (in normal frequency units not angular frequency).

Thus the finesse is... FSR/FWHM =

```
Solve(Finesse == c / (2 L) / (4.5×10^6) /. {L → .728, c → 3×10^8 / 1.03}, Finesse)
```

```
{{Finesse → 44.4539}}
```

The cavity finesse is 44. The Q factor is nothing other than $2\pi\nu$ (Stored Energy)/(Energy Dissipated per Second). If the energy is falling off exponentially then we can say..

```
Qfact = - 2 π ν I0 Exp[-t / τ] / D[I0 Exp[-t / τ], t]
```

```
2 π ν τ
```

Subsituting in for our values we get:

```
Qfact /. {τ → 35×10^-9, ν → 3×10^8 / (532×10^-9)} // N
```

```
1.2401×10^8
```

Q3 30%

Looking at p. 35 of the notes we see that the Gaussian transverse electric field amplitude can be written as:

$$|u(x,y)| = \exp\left(-\frac{(x^2+y^2)}{w^2}\right).$$

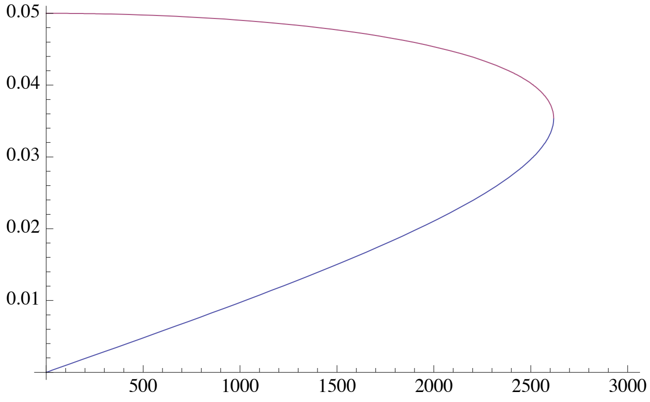
First (as you are told in the question) we can assume that the spot size of the beam can only be as large as 5cm - if it were to be too much bigger then the beam would be sharply chopped off by the edge of the mirrors and hence not be a good approximation to a Gaussian beam - in this case our formaulas would not work - and perhaps more importantly the beam would diffract much more quickly. So the first thing we should work out is the Rayleigh range of the beam:

```
Solve[zR == Pi w0^2/λ /. {λ → 1.5×10^-6, w0 → .05}]
{{zR → 5235.99}}
```

So the Rayleigh range is 5.2km for the beam. In principal, one might think that you could focus the beam exiting the telescope in front of the telecsope and thus limit its expansion by the time it arrived at Rottnest; however, a little calculation shows that this intuition is definitively incorrect. To start with perhaps you can immediately see that the best we could possibly do is to focus to a distance only up to around a 5 km in front of the telescope. Why do I say this? Because, to satisfy my original contraint we are only interested in Gaussian beams that have a spot size of 5cm or less at the telescope - this would mean that we could only focus by a very small amount, and to a distance that is not very far out in front (in comparison to the distance to Rotto). To get a better idea of the answer: below I have plotted those waist sizes that would have a 5cm spot size at the telescope position, as a function of the distance of the waist in front of the telescope (all units are SI).

```
sola = w0 /. Solve[w0 Sqrt[ 1 + (z / (Pi w0^2/λ)) ^2] == .05 ]
Solve::svars : Equations may not give solutions for all "solve" variables. ≫
{w0, w0}
```

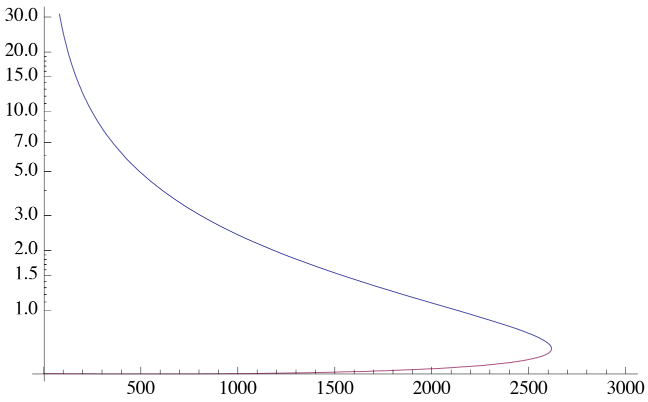
```
Plot[{w0 /. Solve[w0 Sqrt[ 1 + (z / (Pi w0^2/λ)) ^2] == .05 /. {λ → 1.5×10^-6}, w0][[1]],
  solb = w0 /. Solve[w0 Sqrt[ 1 + (z / (Pi w0^2/λ)) ^2] == .05 /. {λ → 1.5×10^-6}, w0][[2]]}, {z,
  0, 3000.7}]
```



We see that there are two possible answers for beams that have a 5cm beam at the telescope for waist positions out to a distance of a little over 2.5km from the telescope. However, there is no possible Gaussian beam that delivers a 5cm beam at the telescope if its waist is more than 2.5km from the telescope. The other way to say this is that if we focus the Gaussian beam exiting the telescope we could force it to have a waist of about 3.5cm at a location a little over 2.5km out from the telescope and then it would expand from this location. As it turns out, however (and I do not think this is obvious without calculation) this focussed solution is far from the best solution to get the smallest waist at Rottnest: in fact the best solution is to essentially collimate the beam out of the telescope (i.e. put the waist at the position of the telescope). I prove this below but it is (of course) not necessary for you to have done this in the assignment. The two solutions below are the two waist sizes as a function of the choice of the waist position, and in the graph below I have plotted the resulting size of the beam at Rottnest as a function of the distance of the waist from the telescope (the lower branch of the solution is that arising from the higher branch in the previous graph above - i.e. the large waist solutions). The smallest waist at Rottnest appears to occur when we choose a large waist and place it exactly at the telescope output. Of course, if you had chosen the small waist solution then it will be a big problem 50km away - if you had chosen the beam that was most nearly collimated you would have done better, and as the calculation shows, the best result is if you chose the beam that is a waist at the telescope location (in fact a more detailed examination finds that the best case scenario is a waist just a few 10s of meters out from the telescope - anyway this can be ignored on the scale of this problem - 50km).

```
solutions = w0 /. Solve[w0 Sqrt[1 + (z / (Pi w0^2/λ))^2] == .05 /. {λ → 1.5 × 10^-6}, w0] ;

LinearLogPlot[
 {(w0 Sqrt[1 + (z1 / (Pi w0^2/λ))^2] //. {z1 → 50 000 - z, λ → 1.5 × 10^-6, w0 → solutions[[2]]}),
  (w0 Sqrt[1 + (z1 / (Pi w0^2/λ))^2] //.
    {z1 → 50 000 - z, λ → 1.5 × 10^-6, w0 → solutions[[4]]})}, {z, 2, 3000}]
```



So assuming a waist at the telescope then the beam size will be :

```
rottosize = w0 Sqrt[1 + (z / (Pi w0^2/λ))^2] /. {w0 → .05, z → 50 000, λ → 1.5 × 10^-6}

0.480076
```

So, the final telescope will only collect power in an area of 3cm so the job becomes one of determining the power in a 3cm diameter area (=xdiameter) at the collection point as compared with the total power in the beam (note the square in the Exp expression because we are interested in the power collected - not the amplitude). Note the 2 π r factor - this is to account for the fact that I am in fact dealing with integrating the power over area (i.e. I have converted the integral to polar co-ordinates and picked up this factor.

$$\mathbf{Integrate}\left[\left(\mathbf{Exp}\left[-\frac{(\mathbf{r}^2)}{\mathbf{w}^2}\right]\right)\mathbf{\hat{}2\ 2\,Pi\,r,\ \{r,\ 0,\ Infinity\},\ Assumptions\ ->\ Re}\left[\mathbf{w}^2\right] > \mathbf{0}\right]$$

$$\frac{\pi\,\mathrm{w}^2}{2}$$

and thus the fraction carried in the middle 3cm is:

$$\mathbf{fracpower = Integrate}\left[2\,\pi\,\mathbf{r}\left(\mathbf{Exp}\left[-\frac{(\mathbf{r}^2)}{\mathbf{w}^2}\right]\right)\mathbf{\hat{}2}\Big/\left(\frac{\pi\,\mathbf{w}^2}{2}\right),\ \mathbf{\{r,\ 0,\ xdiameter / 2\}}\right]\ \mathbf{/.}$$
$$\mathbf{\{w \to rottosize,\ xdiameter \to .03\}}$$

```
0.0019506
```

So if we combine this with the reduction in intensity due to the absorption in the air (a factor of 1/e every 50km). We arrive at the solution:

```
fracpower  Exp[-50 / 50] 100 mW
```

```
0.0717587 mW
```

i.e the solution is 71 μW which is a very detectable amount. In the best case scenario (where we ignore atmospheric turbulance) such a channel can support a many GHz bit data rate. In fact, atmospheric turbulance causes the beam pointing to wander around and make such a link unrealistic over such distances with such small telescopes. As an alternative simpler solution you could have simplified the problem by assuming that a 48cm diameter beam (as it arrives at Rotto) had a pretty flat intensity across the mid few cms. In this case you need to make the connection between a 100mW total power beam and the on-axis value for the intensity (i.e. power/unit area). Equation 4.15 gives the normalized equation as:

$$\sqrt{\frac{2}{\pi}}\ \mathbf{1 / w\ Exp}\left[-\frac{(\mathbf{x}^2+\mathbf{y}^2)}{\mathbf{w}^2}\right]$$

(by normalized I mean the power carried by such a beam is unity when integrated across the entire beam area). The on-axis value of this beam (i.e. the intensity) is (note the squaring to convert amplitude to power units):

$$\left(\sqrt{\frac{2}{\pi}}\ \mathbf{1 / w\ Exp}\left[-\frac{(\mathbf{x}^2+\mathbf{y}^2)}{\mathbf{w}^2}\right]\right)\mathbf{\hat{}2\ /.\ \{x \to 0,\ y \to 0\}}$$

$$\frac{2}{\pi\,\mathrm{w}^2}$$

Thus the on-axis value (the intensity) of a 100mW beam will be 200/(Pi w^2) mW/m^2 and if we incorporate the 3cm diameter area of the collecting telescope and the spot size of the beam, and the exponential reduction in power occuring during transmission we get:

$$\frac{\mathbf{200}}{\pi\,\mathbf{w}^2}\ \pi\ \mathbf{(xdiameter / 2)\ \hat{}\ 2\ Exp[-50 / 50]\ /.\ \{w \to rottosize,\ xdiameter \to .03\}}$$

```
0.0718288
```

where we have nearly achieved exactly the same result as above- thus confirming that our intuition is correct.