Assignment 1:Solutions

1 *a*

ABCD matrices for lens and propagation elements (1 mark for Matrices)

$$\text{lens} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix};$$

$$\text{prop} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix};$$

- **What is the matrix of a flat mirror that is perpendicular to the optical axis? It seems to me that reversing the optical beam direction and handling this within the limited ABCD formalism is quite difficult. My suggestion would be that it would be smarter to undo the effect by eliminating the effect of the reflection. So in this case this means that the beam effectively continues on and then hits a second lens that is positioned a distance away from the first lens by a distance equal to twice the lens-mirror spacing.**

$$\text{mirror} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

Lens followed by *a* free space section followed by *a* lens (2 for putting in correct order, 2 for getting the answer right.)

MatrixForm(lens.prop.prop.lens) // Simplify

$$\begin{pmatrix} 1 - \frac{2\,d}{f} & 2\,d \\ \frac{2\,(d-f)}{f^2} & 1 - \frac{2\,d}{f} \end{pmatrix}$$

One sees one interesting thing here. If the lens-mirror spacing was set equal to the focal length then we have...

$$\begin{pmatrix} 1 - \frac{2\,d}{f} & 2\,d \\ \frac{2\,(d-f)}{f^2} & 1 - \frac{2\,d}{f} \end{pmatrix} \;/.\; d \to f \;//\; \text{MatrixForm}$$

$$\begin{pmatrix} -1 & 2\,f \\ 0 & -1 \end{pmatrix}$$

If we choose a lens with a short enough focal length (meaning that the slope of any incoming ray times this focal length is a small number compared with the perpendicular distance) then the effect of the lens-mirror combination is to invert both the position and slope of the ray. This device is called a cat's-eye in the field as it sends a ray back to you with all the opposite properties (as a animal's eye does when you shine lights into it at night).

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{{-1, 2 f}, {0, -1} }.{x, y}
{-x + 2 f y, -y}
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2. Model the thick lens as two spherical dielectric interfaces between which is placed a dielectric propagation region. (1 mark for model)

Front face of lens (substituting for dielectric constant of air ~ 1) (1 for correct statement of all matrices)

$$\text{front} = \begin{pmatrix} 1 & 0 \\ \frac{n-1}{n\,r1} & \frac{1}{n} \end{pmatrix};$$

Back face of lens

$$\text{back} = \begin{pmatrix} 1 & 0 \\ \frac{1-n}{r2} & n \end{pmatrix};$$

Propagation in the lens

$$\text{transfer} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix};$$

Here is the calculation of the total transfer matrix *N.B.*The order must be right *i.e.*the light hits the front face first (cf.1 *a*) (1 for order, 1 for correct multiplication)

Simplify(total = back.transfer.front) // TraditionalForm

$$\begin{pmatrix} \frac{d\,(n-1)}{n\,r1} + 1 & \frac{d}{n} \\ -\frac{(n-1)\,(d\,(n-1)+n\,(r1-r2))}{n\,r1\,r2} & \frac{-n\,d+d+n\,r2}{n\,r2} \end{pmatrix}$$

```
(back /. {n → 1.25, r2 → - 1}).{1, 0}
```

```
{1., 0.25}
```

Substituting in the material values
(NB : r1 and r2 need to be negative because we want the front surface to be focussing while the back surface should be defocussing) –
(1 mark for correct order, 1 mark for getting signs right)

(actual = total /. {*n* → 1.25, r1 → −1, r2 → 3, *d* → .01}) // TraditionalForm

$$\begin{pmatrix} 0.998 & 0.008 \\ -0.333167 & 0.999333 \end{pmatrix}$$

(b(i)) Let's calculate what happens to the (1,0) vector on application of the thick lens (1 mark for correct application)

(exit = actual.{1, 0}) // TraditionalForm

{0.998, −0.333167}

Set-up an equation to find where it crosses the axis, and solve it: (1 mark for prop, 1 for solving)

prop.exit

```
{0.998 - 0.333167 d, -0.333167}
```

Solve(%⟦1⟧ == 0)

```
{{d → 2.9955}}
```

So this ray will cross the axis $\sim 3\,m$ behind the exit face of the lens

[b(ii)] All rays passing through the optical axis have the form [0,s] where s is the slope of the ray (1 mark for matrices, 1 to point out that it has changed) - this is a simple approach that assumes that going through the midpoint of the front face, the midpoint of the centre of the lens or the midpoint of the back face are all the same. Of course, if it is a thick lens then it is not really clear what one should do. If I prove that it doesn't work for one of these then I guess it shows that this lens does not preserve this property...

total.{0, *s*}

$$\left\{\frac{d\,s}{n}, \frac{\left(n + \frac{d\,(1-n)}{r2}\right)s}{n}\right\}$$

Lets substitute for the lens having thickness = 0

% /. *d* → 0

```
{0, s}
```

So we find that for a zero thickness lens (‘ a thin lens‘) the slope is unchanged. For a lens with non-zero thickness the exit slope differs from the input slope

All rays parallel to the optical axis have the form(*x*, 0) where *x* is the distance from the optical axis

total.{*x*, 0}

$$\left\{\left(1 + \frac{d\,(-1+n)}{n\,r1}\right)x, \left(\frac{(-1+n)\left(n + \frac{d\,(1-n)}{r2}\right)}{n\,r1} + \frac{1-n}{r2}\right)x\right\}$$

To find the point where this ray crosses the optical axis we divide the distance from the optical axis by the slope to get:

Simplify$\left(-\frac{\%[\![1]\!]}{\%[\![2]\!]}\right)$

$$\frac{(d\,(-1+n) + n\,r1)\,r2}{(-1+n)\,(d\,(-1+n) + n\,(r1-r2))}$$

Note that this value is independent of x i.e all parallel rays are heading to the same point on the optical axis - the focal point. (1 mark for matrices, 1 to show not dependent)

What is the focal length of the lens if it is immersed in water? Realize that all rays parallel to the optical axis that are incident on the lens cross the optical axis at the focal point (4 marks):

- **Since I assumed in the very early stage above that the external refractive index was I will need to do the calculation again. Here al is an arbitrary distance after the lens.**

$$\left(\text{waterlens} = \begin{pmatrix} 1 & al \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 \\ \frac{n1-n2}{n1\, r2} & n2/n1 \end{pmatrix} . \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 \\ \frac{n2-n1}{n2\, r1} & \frac{n1}{n2} \end{pmatrix} /. \{n1 \to 1.33,\ n2 \to 1.25,\ r1 \to -1,\ r2 \to 3,\ d \to .01\}\right) // \text{MatrixForm}$$

$$\begin{pmatrix} 1 + 0.064\,(0.01\,(1 + 0.0200501\,al) + 0.93985\,al) + 0.0200501\,al & 1.064\,(0.01\,(1 + 0.0200501\,al) + 0.9398 \\ 0.0802133 & 1.00021 \end{pmatrix}$$

- **So now hit this function with (x,0) and find where it crosses the axis.**

```
outpt = waterlens . {x, 0}
```

```
{(1 + 0.064 (0.01 (1 + 0.0200501 al) + 0.93985 al) + 0.0200501 al) x, 0.0802133 x}
```

- **Take the first term and find when it crosses the axis.**

```
Solve[outpt[[1]] == 0, al]
```

```
{{al → -12.4747}}
```

- **So it has become a lens with a negative focal length - i.e. it is not a convergent lens anymore - rather in water this lens cannot focus.**

(10 marks) Q3 Use the attached figure on the final page for notation. The path difference between the two transmitted rays (which is the same for two reflected rays) is : ACB - AD (remember we must compare the phase of the beams at the two points which will in fact interfere on a screen).

Now AC = CB so ACB = 2 AC = $\frac{2\,d}{\cos\theta}$.

(We must remember that the wavelength is shorter in the medium so the optical path length is 2 d n cos θ.)

Now AD = AB sin θ′ and AB = 2 d tan θ so AD = 2 d tan θ sin θ′
but of course sin θ′ = n sin θ (Snell′ s Law) so AD = 2 dn tan θ sin θ.

The net optical length difference $= 2\,d\,n\left(\frac{1}{\cos\theta} - \tan\theta\sin\theta\right) = \frac{2\,d\,n\,(1-\sin^2\theta)}{\cos\theta} = \frac{2\,d\,n\,(\cos^2\theta)}{\cos\theta} = 2\,d\,n\cos\theta$

$$\text{Thus the net phase difference} = \frac{4\,\pi\,d\,n\cos\theta}{\text{free space wavelength}}$$

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4 (a) Adjacent axial modes occur in a resonator when there is a whole number of half wavelengths between the mirrors
i.e. $L = n\,\lambda/2$
where L is the distance between the mirrors and n is some integer

Rewriting in terms of frequency we get:
$\nu_n = n\,c/(2\,L)$
Therefore frequency spacing between adjacent modes is c/2L. For this resonator:

$$\text{freqspac} == \frac{c}{2\,l} /. \{c \to 3\,10^8/2,\ l \to .001\} // \text{Solve}$$

$$\{\{\text{freqspac} \to 7.5\times10^{10}\}\}$$

i.e. 75GHz (I took into account the refractive index of the material in calculating the speed of light). Since the gain medium has a bandwidth of 1nm we need to calculate the frequency width whch is around (using $\delta\lambda/\lambda \sim -\delta f/f$ which I derived from taking derivatives of the wave equation)

```
1 / 675 == δf / (3 × 10 ^ 8 / (675 × 10 ^ -9)) // Solve // N
```

$\{\{\delta f \to 6.58436 \times 10^{11}\}\}$

there will be around 8 modes falling in the bandwidth depending on whether the modes are symmetrically placed about the centre. (2 mark for 150 MHz, 2 for correct statement of mode number - I am happy with plus or minus 1)

```
6.584362139917695`*^11 / 7.5`*^10
```

8.77915

(b) The F.S.R of the Optical Spectrum Analyzer must be larger than the largest spacing between signals which could possibly enter the analyzer to gain an unambigious output. In this case the largest spacing is about 0.66 THz so the free spectral range of the spectrum analyzer needs to be larger than this:

$$N\left(\text{Solve}\left(\text{freqspac} == \frac{c}{2\,l} \text{ /. } \{c \to 3\,10^8, \text{freqspac} \to 6.58 \times 10 \wedge 11\}, l\right)\right)$$

{{l → 0.000227964}}

The spectrum analyzer must have a mirror spacing of less than 228 microns (2 mark for this, 2 for statment of FSR limits)

(*c*) The smallest details we wish to see are the individual modes at the output of the laser.These modes are 75 GHz apart so lets aim for *a* resolution of 7.50 GHz

$$N\left(\text{Solve}\left(\text{finesse} == \frac{\text{fsr}}{\text{resolution}} \text{ /. } \{\text{fsr} \to 6.58 \times 10 \wedge 11, \text{resolution} \to 7.50\ 10^9\}, \text{finesse}\right)\right)$$

{{finesse → 87.7333}}

What reflectivity is required to achieve this finesse?

$$N\left(\text{Solve}\left(\text{finesse} == \frac{\pi\, r}{1 - r^2} \text{ /. } \{\text{finesse} \to 87.7\}, r\right)\right)$$

{{r → -1.01807}, {r → 0.982249}}

Taking the positive result the (amplitude) reflectivity is 0.982. What is usually termed reflectivity is the power or intensity reflectivity which is the square of this number i.e.

$$(r \text{ /. } \%[[2]])^2$$

0.964814

So the reflectivity required for the resolution needed is ~ 96%. (2 mark for 15MHz, 2 for getting finesse right, 2 for giving r, 2 for understanding difference between power and amplitude reflectivity)