Statistical Mechanics Problems I
(Assessed as extra credit at a value of 3% of total course mark. Due 1 September, 2008.)

1. Questions on scattering.
(a) The total scattering cross section for an electron-air molecule collision is about 10^{-15} cm^2. At what gas pressure will 90% of the electrons emitted from a cathode reach an anode 20 cm away? (Assume that any electron scattered out of the beam does not reach the anode.)

(b) In the Millikan oil-drop experiment, the terminal velocity with which the oil drop falls is inversely proportional to the viscosity of the air. If the temperature of the air increases, does the terminal velocity of the drop increase, decrease, or remain the same?

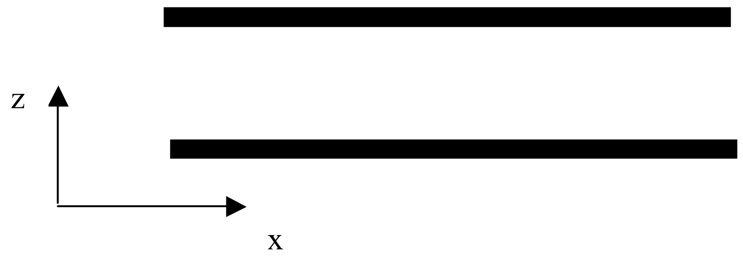
(c) Suppose that the molecules of a gas interact with each other through a radial force F which depends on the intermolecular separation R according to $F = CR^{-s}$, where s is a positive integer and C a constant.

> **i)** Use arguments of dimensional analysis to show how the total scattering cross section of the molecules depends on their relative speed V. The cross section should depend on V, molecular mass m, and C.
>
> **ii)** How does the coefficient of viscosity of this gas depend on the temperature T?

(d) A satellite, in the form of a cube of edge length L, moves through outer space with a velocity V parallel to one of its edges. The surrounding gas consists of molecules of mass m at a temperature T. The number n of molecules per unit volume is very small, so that the mean free path of the molecules is much larger than L. Assuming that collisions of the molecules with the satellite are elastic, calculate the mean retarding force exerted on the satellite by the gas. Assume that V is small compared to the mean speed of the gas molecules. After how long a time will the velocity of the satellite be reduced to half its original value?

2. Questions on the Boltzmann Equation.

(a) Suppose that a viscous fluid is sandwiched by two plates as shown:



The top plate moves with velocity u_x in the x direction and is a function of z. The effect of collisions in the gas is to produce a local equilibrium distribution of velocities relative to the gas moving with the mean velocity u_x at position z. The equilibrium distribution is defined by

$$f_o(v_x - u_x, v_y, v_z) = n\left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-\frac{m[(v_x - u_x)^2 + v_y{}^2 + v_z{}^2]}{2k_B T}}.$$

i) Show that f_o satisfies the steady state Boltzmann equation in the relaxation time approximation if u_x is <u>not</u> a function of z (i.e., independent of position).

ii) Show that if u_x depends on z, the steady state Boltzmann equation reduces to

$$v_z \frac{\partial f}{\partial z} = -\frac{f - f_o}{\tau}.$$

iii) Assuming that the gradient term is sufficiently small, show that

$$f = f_o + \tau v_z \frac{\partial f_o}{\partial U_x} \frac{\partial u_x}{\partial z}$$

where $U_x = v_x - u_x$.

(b) Argue that the equilibrium condition $f' f_2{}' = f f_2$ can be satisfied only by an expression of the form

$$\ln f = A + B_x m v_x + B_y m v_y + B_z m v_z + C\left(\frac{1}{2} m v^2\right).$$

Show that this implies that f must be the Maxwellian velocity distribution for a gas (whose mean velocity does not necessarily vanish).

(c) Derive a law for the conservation of momentum in fluids by using f to construct the average $\langle m\mathbf{v}\rangle$.

i) Use the Boltzmann equation together with the conservation of mass to show that the average of each component of the momentum, $\langle m v_\gamma \rangle$, obeys

$$\frac{\partial}{\partial t}\rho\langle v_\gamma\rangle + \frac{\partial}{\partial x_\alpha}\rho\langle v_\alpha v_\gamma\rangle = \rho\frac{F_\gamma}{m}.$$

(Notation: repeated subscripts imply summation.)

ii) With the definition of a relative velocity $\mathbf{U} = \langle \mathbf{v} \rangle - \mathbf{v}$, and the "pressure tensor" $P_{\alpha\beta} = \rho\langle U_\alpha U_\gamma \rangle$, derive the Euler equation of motion for hydrodynamics:

$$\frac{\partial}{\partial t}\rho u_\gamma + \frac{\partial}{\partial x_\alpha}\rho u_\alpha u_\gamma = -\frac{\partial P_{\alpha\beta}}{\partial x_\alpha} + \rho\frac{F_\gamma}{m}.$$

iii) Use conservation of mass and the definition $\frac{d}{dt}u_\gamma = \frac{\partial}{\partial t}u_\gamma + u_\alpha \frac{\partial}{\partial x_\alpha}u_\gamma$ to write Euler's equation of motion in the form

$$\rho\frac{d}{dt}u_\gamma = -\frac{\partial}{\partial x_\alpha}P_{\alpha\gamma} + \rho\frac{F_\gamma}{m}.$$

Give a physical interpretation of each term in this equation. Explain how this equation describes changes in momentum of any element of the fluid due to stress and pressure from the surrounding fluid (and any external forces).