STAT MECH SOLUTIONS : Problems III

(a) Let $n = \langle n_i \rangle$, the mean occupation.

Then

$$H = -\epsilon \sum_{\langle i,j \rangle} n_i n_j$$

near neighbours

$$= -\epsilon \sum_{\langle i,j\rangle} (n_i - n)(n_j - n) - \epsilon \sum_{\langle i,j \rangle} \left(-n^2 + n(n_i + n_j)\right)$$

The mean field H is

$$H_{mf} = -\epsilon \sum_{\langle i,j\rangle} \left(-n^2 + n(n_i + n_j)\right)$$

$$= \epsilon n^2 \sum_{\langle i,j \rangle} - \epsilon n \sum_{\langle i,j\rangle} (n_i + n_j)$$

$$= \epsilon n^2 \frac{1}{2} \sum_i \sum_{i+\delta} - 2\epsilon n \frac{1}{2} \sum_i \sum_{i+\delta} n_i$$

factor to not double count

index to n.n.

$$= \epsilon n^2 \frac{1}{2} \sum_i 6 - \epsilon n \sum_i 6 n_i$$

$$\boxed{= \sum_i \left\{ 3\epsilon n^2 - 6\epsilon n n_i \right\}}$$

$\boxed{(b)}$ i) $$\frac{\partial f}{\partial \varphi} = -\alpha(T_c - T)\varphi + b\varphi^3$$

for φ independent of x, Then

$$\frac{\partial f}{\partial \varphi} = 0 \Rightarrow \boxed{\varphi_0 = \pm\sqrt{\frac{\alpha(T_c-T)}{b}}}$$

ii) $$\lambda \frac{\partial^2}{\partial x^2} \varphi_0 \tanh(x/L) = \lambda \varphi_0 \frac{\partial}{\partial x}\left(\frac{\frac{1}{L}\cosh x/L}{\cosh x/L}\right.$$

$$\left. - \frac{\sinh \frac{x}{L}}{\cosh^2 \frac{x}{L}} \frac{1}{L}\sinh\frac{x}{L}\right) = -\frac{\lambda}{L}\varphi_0 \frac{\partial}{\partial x}\frac{\sinh^2 x/L}{\cosh^2 x/L}$$

$$= -\frac{\lambda}{L}\varphi_0\left(\frac{2\sinh\frac{x}{L}\cosh\frac{x}{L}}{L\cosh^2 x/L} - 2\frac{\sinh^2 x/L}{\cosh^3 x/L}\frac{\sinh\frac{x}{L}}{L}\right)$$

$$= -\frac{2\lambda}{L^2}\varphi_0\left(\tanh\frac{x}{L} - \tanh^3 x/L\right)$$

Substitution into $-\alpha(T_c-T)\varphi + b\varphi^3$ leads to the conditions

$$-\frac{2\lambda}{L^2}\varphi_0 = -\alpha(T_c-T)\varphi_0$$

$$-\frac{2\lambda}{L^2}\varphi_0 = b\varphi_0^3$$

$$\Rightarrow \quad L^2 = \frac{2\lambda}{\alpha(T_c-T)}$$

and $$\varphi_0^2 = -\frac{2\lambda}{bL^2} = -\frac{\alpha(T_c-T)}{b}$$