STAT MECH SOLUTIONS : Problems II

1a Langevin equation of form

$$m \frac{dv}{dt} = -\alpha v + F$$

mass soot   friction   random force

Expect α to be function of viscosity η and particle radius r. Dimensional analysis gives

$$[\alpha]_{\text{units}} = \text{kg}/\text{s}$$

$$\Rightarrow \quad \alpha \propto \eta r$$

since $\quad [\eta]_{\text{units}} = \text{kg}/(\text{m}\cdot\text{s})$.

Set $\alpha = a \eta r$ where a is a constant we don't know.

From notes on Langevin equation, $\frac{\alpha}{m} = \frac{1}{\tau}$

and

$$D = \frac{k_B T}{m} \tau = \frac{k_B T}{\alpha}$$

But $\quad \langle x^2 \rangle = 6 D t = \frac{6 k_B T}{\alpha} t$

With $k_B = R/N_A$,

$$\langle x^2 \rangle = \frac{6RT}{N_A \alpha} t$$

$$\Rightarrow \quad N_A = \frac{6RT}{\langle x^2 \rangle \alpha} t = 6 \frac{RTt}{\langle x^2 \rangle a \eta r}$$

$$= \frac{6 \left(8.314 \frac{cm^3\, MPa}{K\, mol}\right)(273.15+18.8)\, 110\, s}{(3.3 \times 10^{-8}\, cm^2)\, a\, (0.0278\, g/cm\, s)\, (0.4 \times 10^{-4}\, cm)}$$

$$= \frac{1}{a} \cdot 6.6 \times 10^{17} \times 10^{7}\, mol^{-1}$$

$$\stackrel{!}{=} \boxed{\frac{1}{a}\, 6.6 \times 10^{24}\, mol^{-1}}$$

Comparison to known value for N_A ($6 \times 10^{23}\, mol^{-1}$) gives

$$a = 11$$

In fact, a is given by Stokes law for motion of a sphere in a viscous liquid as $a = 6\pi$.

1b i) Let $F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, e^{i\omega t} F_\omega$

$$v(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, e^{i\omega t} v_\omega$$

Then $\frac{dv}{dt} = -\gamma v + \frac{1}{m} F$

becomes

$$i\omega v_\omega = -\gamma v_\omega + \frac{1}{m} F_\omega$$

$$\Rightarrow \boxed{v_\omega = \frac{F_\omega / m}{\gamma + i\omega}}$$

ii) The Wiener-Khintchine relation in complex Fourier transforms is

$$\Omega(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ds \; e^{-i\omega s} \langle v(0) v(s) \rangle$$

$$= \frac{\langle v^2(0)\rangle}{2\pi}\int_{-\infty}^{\infty} ds \, e^{-i\omega s} \, e^{-\gamma |s|}$$

$$= \frac{\langle v^2(0)\rangle}{2\pi}\left\{ \int_{0}^{\infty} ds \, e^{-i\omega s} \, e^{-\gamma s} + \int_{-\infty}^{0} ds \, e^{-i\omega s} \, e^{\gamma s} \right\}$$

$$= \frac{\langle v^2(0)\rangle}{2\pi}\left\{ -\frac{1}{\gamma + i\omega}(-1) + \frac{1}{\gamma - i\omega}(1) \right\}$$

$$= \frac{\langle v^2(0)\rangle}{2\pi}\left[\frac{1}{\gamma+i\omega} + \frac{1}{\gamma-i\omega}\right]$$

$$= \frac{\langle v^2(0)\rangle}{2\pi}\left(\frac{2\gamma}{\gamma^2+\omega^2}\right)$$

$$\boxed{= \frac{k_B T}{2\pi m}\left(\frac{2\gamma}{\gamma^2+\omega^2}\right)}$$

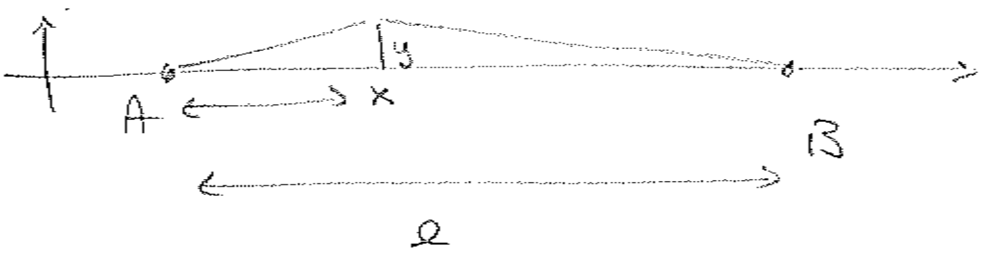
iii) Form $\langle V_\omega^* V_\omega\rangle = \dfrac{\frac{1}{m^2}\langle F_\omega^* F_\omega\rangle}{\gamma^2+\omega^2}$

But $\langle \Omega(\omega)\rangle = \langle V_\omega^* V_\omega\rangle$, so

$$\frac{\frac{1}{m^2}\langle F_\omega^* F_\omega\rangle}{\gamma^2+\omega^2} = \frac{k_B T}{2\pi m}\cdot\frac{2\gamma}{\gamma^2+\omega^2}$$

$$\Rightarrow \boxed{\langle F_\omega^* F_\omega\rangle = \frac{m\gamma}{\pi} k_B T}$$

$\boxed{2a}$



From analysis of forces at x, we know that force on element 'dx of' string at x is

$$df = \left(F \frac{\partial^2}{\partial x^2} y \right) dx$$

Summed over string,

$$f = F \int_0^l dx \frac{\partial^2}{\partial x^2} y = F \left[\frac{\partial y}{\partial x}\Big|_l - \frac{\partial y}{\partial x}\Big|_0 \right].$$

From geometry, have slopes:

$$\frac{\partial y}{\partial x}\Big|_0 = \frac{y}{x} \quad \& \quad \frac{\partial y}{\partial x}\Big|_l = \frac{y}{x-l}$$

so

$$f = Fy \left(\frac{1}{x} - \frac{1}{x-l} \right) = - \frac{Fyl}{x(x-l)}$$

The energy is equal to work done in changing y:

$$\mathcal{E} = \int_0^y d\tilde{y} \frac{Fl}{x(l-x)} \tilde{y} = \frac{1}{2} Fl \frac{y^2}{x(l-x)}$$

Define the thermal average in terms of probability $P(\varepsilon)$. Then we can write

$$\langle \varepsilon \rangle = \frac{1}{2} k_B T = \frac{1}{2} F \ell \; \frac{\langle y^2 \rangle}{x(\ell - x)}$$

or

$$\boxed{\langle y^2 \rangle = \frac{k_B T}{F \ell} \, x(\ell - x)}$$

$\boxed{2b}$ Evaluate

$$\Omega(f) = \int_{-\infty}^{\infty} ds\; K(0)\, e^{-\alpha^2 s^2} \cos(\omega s)\, e^{-i f s}$$

$$= \int_{-\infty}^{\infty} ds\; K(0)\, e^{-\alpha^2 s^2} \cos(\omega s)\left(\cos fs + i \sin fs\right)$$

$$= \int_{-\infty}^{\infty} ds\; K(0)\, e^{-\alpha^2 s^2} \cos \omega s \cos f s$$

$$= 2\int_{0}^{\infty} ds\; K(0)\, e^{-\alpha^2 s^2} \frac{1}{2}\left(\cos(s(\omega+f)) + \cos(s(\omega-f))\right)$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{\alpha^2}}\, K(0)\left(e^{-(\omega+f)^2/(4\alpha^2)} + e^{-(\omega-f)^2/(4\alpha^2)}\right)$$

$$\left(\text{using } 2\int_0^{\infty} \cos(kx)\, e^{-\gamma k^2} dk = \sqrt{\frac{\pi}{\gamma}}\, e^{-x^2/4\gamma}\right)$$

i) $\lim_{\alpha\to 0} \Omega = \lim_{a\to\infty} \frac{1}{2}\sqrt{\pi a^2}\, K(0)\left[e^{-\frac{a^2}{4}(\omega+f)^2} + e^{-\frac{a^2}{4}(\omega-f)^2}\right]$

$\to 0$ if $\omega \pm f \neq 0$. Otherwise a sharp peak at $\omega = f$ & $\omega = -f$

ii) $\lim_{\omega\to 0} \Omega = \sqrt{\frac{\pi}{\alpha^2}}\, K(0)\; e^{-\frac{f^2}{4\alpha^2}}$

iii) $\lim_{\substack{\alpha\to 0\\ \omega\to 0}} \Omega \to 0$ unless $f = 0$

The spectral density is largest for $f = \pm\omega$, and the breadth of frequencies around ω are determined by α.