Statistical Mechanics Problems I
(Assessed as extra credit at a value of 3% of total course mark. Due 1 September, 2008.)

1. Questions on scattering.
(a) The total scattering cross section for an electron-air molecule collision is about 10^{-15} cm^2. At what gas pressure will 90% of the electrons emitted from a cathode reach an anode 20 cm away? (Assume that any electron scattered out of the beam does not reach the anode.)

(b) In the Millikan oil-drop experiment, the terminal velocity with which the oil drop falls is inversely proportional to the viscosity of the air. If the temperature of the air increases, does the terminal velocity of the drop increase, decrease, or remain the same?

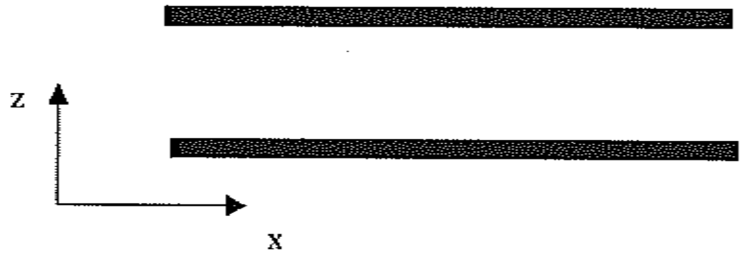
(c) Suppose that the molecules of a gas interact with each other through a radial force F which depends on the intermolecular separation R according to $F = CR^{-s}$, where s is a positive integer and C a constant.

> i) Use arguments of dimensional analysis to show how the total scattering cross section of the molecules depends on their relative speed V. The cross section should depend on V, molecular mass m, and C.
>
> ii) How does the coefficient of viscosity of this gas depend on the temperature T?

(d) A satellite, in the form of a cube of edge length L, moves through outer space with a velocity V parallel to one of its edges. The surrounding gas consists of molecules of mass m at a temperature T. The number n of molecules per unit volume is very small, so that the mean free path of the molecules is much larger than L. Assuming that collisions of the molecules with the satellite are elastic, calculate the mean retarding force exerted on the satellite by the gas. Assume that V is small compared to the mean speed of the gas molecules. After how long a time will the velocity of the satellite be reduced to half its original value?

2. Questions on the Boltzmann Equation.

(a) Suppose that a viscous fluid is sandwiched by two plates as shown:



The top plate moves with velocity u_x in the x direction and is a function of z. The effect of collisions in the gas is to produce a local equilibrium distribution of velocities relative to the gas moving with the mean velocity u_x at position z. The equilibrium distribution is defined by

$$f_o(v_x - u_x, v_y, v_z) = n\left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-\frac{m[(v_x-u_x)^2+v_y^2+v_z^2]}{2k_B T}}.$$

i) Show that f_o satisfies the steady state Boltzmann equation in the relaxation time approximation if u_x is <u>not</u> a function of z (i.e., independent of position).

ii) Show that if u_x depends on z, the steady state Boltzmann equation reduces to

$$v_z \frac{\partial f}{\partial z} = -\frac{f - f_o}{\tau}.$$

iii) Assuming that the gradient term is sufficiently small, show that

$$f = f_o + \tau v_z \frac{\partial f_o}{\partial U_x}\frac{\partial u_x}{\partial z}$$

where $U_x = v_x - u_x$.

(b) Argue that the equilibrium condition $f'f_2' = ff_2$ can be satisfied only by an expression of the form

$$\ln f = A + B_x m v_x + B_y m v_y + B_z m v_z + C\left(\frac{1}{2}mv^2\right).$$

Show that this implies that f must be the Maxwellian velocity distribution for a gas (whose mean velocity does not necessarily vanish).

(c) Derive a law for the conservation of momentum in fluids by using f to construct the average $\langle m\mathbf{v}\rangle$.

i) Use the Boltzmann equation ~~together with the conservation of mass~~ to show that the average of each component of the momentum, $\langle mv_\gamma\rangle$, obeys

$$\frac{\partial}{\partial t}\rho\langle v_\gamma\rangle + \frac{\partial}{\partial x_\alpha}\rho\langle v_\alpha v_\gamma\rangle = \rho\frac{F_\gamma}{m}.$$

(Notation: repeated subscripts imply summation.)

ii) With the definition of a relative velocity $\mathbf{U} = \langle \mathbf{v} \rangle - \mathbf{v}$, and the "pressure tensor" $P_{\alpha\beta} = \rho \langle U_\alpha U_\gamma \rangle$, derive the Euler equation of motion for hydrodynamics:

$$\frac{\partial}{\partial t} \rho u_\gamma + \frac{\partial}{\partial x_\alpha} \rho u_\alpha u_\gamma = -\frac{\partial P_{\alpha\beta}}{\partial x_\alpha} + \rho \frac{F_\gamma}{m}.$$

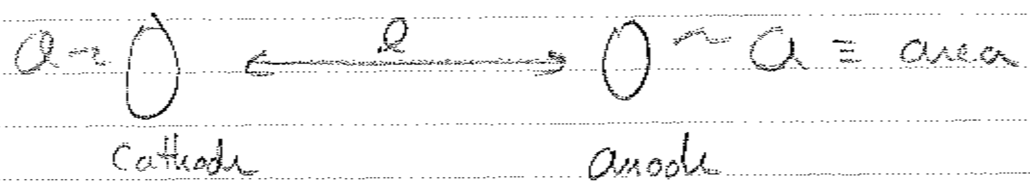
iii) Use conservation of mass and the definition $\frac{d}{dt} u_\gamma = \frac{\partial}{\partial t} u_\gamma + u_\alpha \frac{\partial}{\partial x_\alpha} u_\gamma$ to write Euler's equation of motion in the form

$$\rho \frac{d}{dt} u_\gamma = -\frac{\partial}{\partial x_\alpha} P_{\alpha\gamma} + \rho \frac{F_\gamma}{m}.$$

Give a physical interpretation of each term in this equation. Explain how this equation describes changes in momentum of any element of the fluid due to stress and pressure from the surrounding fluid (and any external forces).

Solutions - Problems I STAT MECH

1 a



10% $\bar{e}$ scattered $\Rightarrow$

$$0.1 \left(\frac{\text{no. incident}}{\text{time}}\right) = \sigma_0 \left(\frac{\text{no. incident}}{\text{area} \cdot \text{time}}\right)\left(\text{no. targets}\right)$$

so

$$0.1 = \sigma_0 \left(\frac{\text{no. targets}}{a}\right)$$

But if gas ideal, no. targets $\equiv N = \frac{PV}{k_B T}$

also,

$$V = a\ell$$

$$\Rightarrow \quad \boxed{0.1 = \sigma_0 \frac{P}{k_B T} \ell}$$

At room temp $T = 300\,K$, so

$$P = \frac{k_B T}{10\, \sigma_0 \ell} = \frac{(1.381 \times 10^{-23}\ J/K) \cdot 3 \times 10^{2}\ K}{10\ (10^{-19}\ m^2)(0.2\ m)}$$

$$\boxed{= 2.1 \times 10^{-2}\ Pa}$$

$\boxed{1b}$ From notes, $\eta \propto \frac{\langle v \rangle}{\sigma_0}$. T dependence from $\langle v \rangle \approx \sqrt{T}$ since by equipartition $\langle v^2 \rangle \propto T$.

Thus, if
$$V_{terminal} \propto \frac{1}{\eta} \Rightarrow V_{terminal} \propto T^{-1/2}.$$

Therefore, $\boxed{\text{increase } T \Rightarrow \text{decrease } V_{terminal}}$.

$\boxed{1c}$

i) $[\sigma_0]_u = m^2$ (units) If $\sigma_0(v, M, C)$ where $[C]_u = N \cdot m^s$ & $[M] = kg$, then

require $\boxed{\sigma_0(v, M, C) \propto \left[\frac{C}{Mv^2}\right]^{\frac{2}{s-1}}}$

since $\left[\frac{C}{Mv^2}\right]_u = \frac{kg}{kg} \frac{m}{s^2} \frac{s^2}{m^2} m^{+s} = m^{s-1}$

and we require
$$\left[\left[m^{s-1}\right]^{\alpha}\right]_u = m^2 \Rightarrow \alpha = \frac{2}{s-1}$$

ii) Viscosity depends on average relative velocity v according to

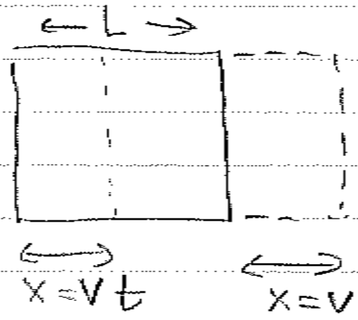
$$\eta \propto \frac{v}{\sigma_0}$$

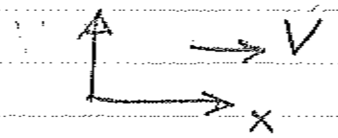
Then here, $\eta \propto v / (v^{-2})^{\frac{2}{s-1}} = v^{\left(\frac{s-3}{s-1}\right)}$

If $v \propto T^{1/2}$, $\boxed{\eta \propto T^{\left(\frac{s-3}{2s-2}\right)}}$

---

1 d   In time t, cube moves $x = V t$

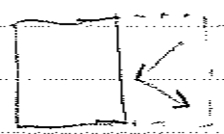




Let N = no. molecules in volume $L^2 x$ $= n L^2 x$

In cube's frame of reference, no. collisions in time t are:

front : $n L^2 (V + V_x) t$



$\leftarrow V_x + V \equiv$ relative gas vel.

back : $n L^2 (V_x - V) t$

$V_x' - V \equiv$ relative gas vel,

The no. collisions on front is greater than back by amount:

$$\Delta N = 2 n L^2 V_x \, t$$

The average force is then in $-x$ direction, with magnitude

$$|F| = \Delta N \left(\begin{array}{c} \text{change in momentum of} \\ \text{gas per collision} \\ \text{per time} \end{array} \right)$$

$$= \Delta N \; \frac{2 m V_x}{t}$$

$$= 4 n L^2 m V_x^2$$

$$= 8 n L^2 \left(\tfrac{1}{2} m V_x^2 \right)$$

$$= 8 n L^2 k_B T$$

The equation of motion of cube is then:

$$M \frac{d}{dt} V = - |F|$$

$$\Rightarrow \quad V = - \frac{8 n L^2 k_B T}{M} t + V(0)$$

$V(t) = \frac{1}{2} V(0)$ when

$$\boxed{t = \frac{M}{16 n L^2 k_B T} V(0)}$$

2a Boltzmann Eq in relaxation time approx:

$$\frac{\partial}{\partial t} f + \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} + \dot{\vec{v}} \cdot \frac{\partial f}{\partial \vec{v}} = -\frac{1}{\tau}(f - f_0)$$

i) if $f = f_0$, $\frac{\partial}{\partial t} f_0 = 0$ and Eq becomes

$$\vec{v} \cdot \frac{\partial f_0}{\partial \vec{r}} + \dot{\vec{v}} \cdot \frac{\partial f_0}{\partial \vec{v}} \stackrel{?}{=} 0$$

With $\vec{u}$ not a function of position, $\frac{\partial}{\partial \vec{r}} f_0 = 0$

With no forces present, $\dot{\vec{v}} = 0$. Thus

f_0 satisfies Boltzmann Eq.

ii) With no forces, only term on left side in steady state is

$$\vec{v} \cdot \frac{\partial}{\partial \vec{r}} f$$

If only u_x depends on z,

$$\vec{v} \cdot \frac{\partial}{\partial \vec{r}} f = \vec{v} \cdot \hat{z} \frac{\partial}{\partial z} f = v_z \frac{\partial}{\partial z} f$$

$$\Rightarrow \quad v_z \frac{\partial}{\partial z} f = -\frac{1}{\tau}(f - f_0)$$

iii) Let $f = f_0 + \mathcal{f}$ where $\mathcal{f}$ is small. Then

$$v_z \frac{\partial}{\partial z}(f_0 + \mathcal{f}) = -\frac{1}{\tau} \mathcal{f}$$

Because $f_0(u_x, v_y, v_z)$, and $u_x(z)$, have

$$\frac{\partial}{\partial z} f_0 = \left(\frac{\partial}{\partial u_x} f_0\right) \frac{\partial u_x}{\partial z} = \left(\frac{\partial f_0}{\partial u_x}\right)\left(\frac{\partial u_x}{\partial z}\right)$$

then

$$v_z \left[\left(\frac{\partial f_0}{\partial u_x}\right)\left(\frac{\partial u_x}{\partial z}\right) + \frac{\partial}{\partial z} f \right] = -\frac{1}{\tau} f$$

$$\Longrightarrow \quad f = f(0)\, e^{-\frac{z}{\tau v_z}} + \tau v_z \left(\frac{\partial f_0}{\partial u_x}\right)\left(\frac{\partial u_x}{\partial z}\right)$$

If $\tau v_z \gg$ than plate separation, neglect first term for relevant values of z. Then

$$\boxed{f \simeq f_0 + \tau v_z \left(\frac{\partial f_0}{\partial u_x}\right)\left(\frac{\partial u_x}{\partial z}\right)}$$

$\boxed{2b}$ At equilibrium,

$$f(v')f(v_2') = f(v)f(v_2)$$

We require f to have form of a probability distribution since it satisfies steady state Boltzmann Equation. Thus ~~ln f~~ is a function of v. Write as a power series:

$$\ln f = a + \vec{B}\cdot\vec{V} + \vec{V}\cdot\overleftrightarrow{C}\,\vec{V} + \dots$$

But from Boltzmann collision term,

$$\ln f(v') + \ln f(v_2') = \ln f(v) + \ln f(v_2)$$

Only combination of velocities that conserve momentum and energy are allowed:

$$\tilde{A} + \tilde{B}_x m(V_x' + V_{2x}') + \tilde{B}_y m(V_y' + V_{2y}') + \tilde{B}_z m(V_z' + V_{2z}')$$
$$+ \tilde{C}\left(\tfrac{1}{2}mV'^2 + \tfrac{1}{2}mV_2'^2\right)$$

$$= \tilde{B}_x m(V_x + V_{2x}) + \tilde{B}_y m(V_y + V_{2y}) + \tilde{B}_z m(V_z + V_{2z})$$
$$+ \tilde{C}\left(\tfrac{1}{2}mV^2 + \tfrac{1}{2}mV_2^2\right) + \tilde{A}$$

arbitrary constant allowed

The constant $\tilde{A}$ can be a function of mean velocities $\vec{u}$ such that

$$\ln f = \tilde{A} + \frac{\tilde{B}}{2}m\left[(v_x-u_x)^2 + (v_y-u_y)^2 + (v_z-u_z)^2\right]$$

Therefore f, as a solution of equilibrium Boltzmann Equation, has the form of Maxwell distribution for nonzero mean ($\tilde{C} = \frac{1}{\ }$).

$\boxed{2c}$ $\quad \langle \chi \rangle = \int d^3v \, f \, \chi$; Here $\chi = mV_\gamma$

i) Suppose no collisions & average χ times Boltzmann equation:

$$\int d^3v \, \chi \left\{ \frac{\partial}{\partial t} f + \vec{v} \cdot \frac{\partial}{\partial \vec{r}} f + \frac{\vec{F}}{m} \cdot \frac{\partial f}{\partial \vec{v}} \right\} = 0$$

Use repeated indicies => summation. Then

$$\int d^3v (mV_\gamma) \left\{ \frac{\partial}{\partial t} f + V_\alpha \frac{\partial}{\partial x_\alpha} f + \frac{F_\alpha}{m} \frac{\partial f}{\partial V_\alpha} \right\} = 0$$

Use (where $n \equiv$ no. density)

$$\int d^3v \, (mV_\gamma) \frac{\partial}{\partial t} f = \int d^3v \left[\frac{\partial}{\partial t} (mV_\gamma f) - f \left(\frac{\partial}{\partial t} mV_\gamma \right) \right]$$

$$= \frac{\partial}{\partial t} n \langle mV_\gamma \rangle - n \langle \frac{\partial}{\partial t} mV_\gamma \rangle$$

$$\int d^3v \, (mV_\gamma) V_\alpha \frac{\partial}{\partial x_\alpha} f = \int d^3v \left[\frac{\partial}{\partial x_\alpha} (f \, mV_\alpha V_\gamma) - f \left(\frac{\partial}{\partial x_\alpha} mV_\alpha V_\gamma \right) \right]$$

$$= \frac{\partial}{\partial x_\alpha} n \langle mV_\alpha V_\gamma \rangle - n \langle m \frac{\partial}{\partial x_\alpha} V_\alpha V_\gamma \rangle$$

Also, assume $\vec{F}$ independent of $\vec{v}$, and use

$$\int d^3v \, (mV_\gamma) \frac{F_\alpha}{m} \frac{\partial f}{\partial V_\alpha} = \int d^3v \left[\frac{\partial}{\partial V_\alpha} (V_\gamma F_\alpha f) - f F_\alpha \left(\frac{\partial V_\gamma}{\partial V_\alpha} \right) \right]$$

$=$

$$= V_\gamma F_\alpha f \Big|_{V_\alpha=-\infty}^{V_\alpha=\infty} - \int d^3v \, f \, F_\alpha \, \delta_{\alpha\gamma}$$

since $\frac{\partial V_\gamma}{\partial V_\alpha} = 0$ if $\gamma \neq \alpha$. Then since $f \to 0$ at $\pm\infty = V_\alpha$,

$$\int d^3v \, (mV_\gamma) \frac{F_\alpha}{m} \frac{\partial f}{\partial V_\alpha} = - F_\gamma \int d^3v \, f$$

$$= - F_\gamma \, n$$

And finally,

$$\underbrace{\frac{\partial}{\partial t} n \langle mV_\gamma \rangle - n \langle \frac{\partial}{\partial t} mV_\gamma \rangle + \frac{\partial}{\partial x_\alpha} n \langle mV_\alpha V_\gamma \rangle}_{- \langle m \frac{\partial}{\partial x_\alpha} V_\alpha V_\gamma \rangle} - F_\gamma n = 0$$

The mass density ρ is nm. Also, write

$$u_\alpha = \langle V_\alpha \rangle$$

Then the equation becomes

$$\frac{\partial}{\partial t} \rho u_\gamma + \frac{\partial}{\partial x_\alpha} \rho \langle V_\alpha V_\gamma \rangle =$$

$$\rho \langle \frac{\partial}{\partial t} V_\gamma \rangle + \rho \langle \frac{\partial}{\partial x_\alpha} V_\alpha V_\gamma \rangle + \rho \frac{F_\gamma}{m}$$

We assume that variables t, $\vec{r}$ and $\vec{v}$ are independent. Then $\frac{\partial}{\partial t} V_\gamma = 0$ and $\frac{\partial}{\partial x_\alpha} V_\beta = 0$. Then

$$\boxed{\frac{\partial}{\partial t} \rho u_\gamma + \frac{\partial}{\partial x_\alpha} \rho \langle V_\alpha V_\gamma \rangle = \rho \frac{F_\gamma}{m}}$$

ii) Define $U_\alpha = v_\alpha - v_\alpha$. Then

$$V_\alpha V_\gamma = (u_\alpha - U_\alpha)(u_\gamma - U_\gamma)$$

$$= u_\alpha u_\gamma - u_\alpha U_\gamma - u_\gamma U_\alpha + U_\alpha U_\gamma$$

Averaging,

$$\langle v_\alpha v_\gamma \rangle = u_\alpha u_\gamma - u_\alpha \langle U_\gamma \rangle - u_\gamma \langle U_\alpha \rangle + \langle U_\alpha U_\gamma \rangle$$

$$= u_\alpha u_\gamma + \langle U_\alpha U_\gamma \rangle$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} \rho u_\gamma + \frac{\partial}{\partial x_\alpha} \rho u_\alpha u_\gamma = -\frac{\partial}{\partial x_\alpha} \rho \langle U_\alpha U_\gamma \rangle + \rho \frac{F_\gamma}{m}}$$

iii) With uniform and stationary ρ, and $P_{\alpha\beta} = \rho \langle U_\alpha U_\beta \rangle$

$$\rho\left(\frac{\partial}{\partial t} u_\gamma + \frac{\partial}{\partial x_\alpha} u_\alpha u_\gamma\right) = -\frac{\partial}{\partial x_\alpha} P_{\alpha\gamma} + \rho \frac{F_\gamma}{m}$$

$$\rho\left(\frac{\partial}{\partial t} u_\gamma + u_\alpha \frac{\partial}{\partial x_\alpha} u_\gamma\right) = -\frac{\partial}{\partial x_\alpha} P_{\alpha\gamma} + \rho \frac{F_\gamma}{m}$$

since $\rho \frac{\partial}{\partial x_\alpha} u_\alpha = -\frac{\partial}{\partial t} \rho = 0$, by conservation of mass. Finally,

$$\frac{d}{dt} u_\gamma = \frac{\partial}{\partial t} u_\gamma + \left(\frac{\partial}{\partial x_\alpha} u_\gamma\right)\left(\frac{\partial x_\alpha}{\partial t}\right)$$

$$\Rightarrow \boxed{\rho \frac{d}{dt} u_\gamma = -\frac{\partial}{\partial x_\alpha} P_{\alpha\gamma} + \rho \frac{F_\gamma}{m}}$$

The first term is time rate of change of γ momentum component.

The second term is the force associated with stress coupling γ and α components in the fluid.

The third term is an external force.