

**2008 THIRD YEAR STATISTICAL MECHANICS TEST B**

**THE FOLLOWING INFORMATION MAY BE USEFUL**

$$\hbar = 1.05 \times 10^{-34} \text{ J s} = 6.6 \times 10^{-16} \text{ eV s}, 1 \text{ eV} = 1.60 \times 10^{-19} \text{ J}$$

$$\text{electron mass } m_e = 9.11 \times 10^{-31} \text{ kg}, \text{ neutron mass } m_n = 1.67 \times 10^{-27} \text{ kg}$$

$$k_B = 1.381 \times 10^{-23} \text{ J/K}, N_A = 6.022 \times 10^{23} \text{ mol}^{-1}$$

$$e^{iax} = \cos(ax) + i\sin(ax), \cos(ax) = \frac{1}{2}(e^{iax} + e^{-iax}), \sin(ax) = \frac{1}{2i}(e^{iax} - e^{-iax})$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \dots \quad N! \approx N^N e^{-N} \sqrt{2\pi N} \quad \sum_{n=0}^N x^n = \frac{1-x^{N+1}}{1-x}$$

$$\int_0^{\infty} e^{-\lambda x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{\lambda}} \quad \int_0^{\infty} x e^{-\lambda x^2} dx = \frac{1}{2\lambda} \quad \int_0^{\infty} x^2 e^{-\lambda x^2} dx = \frac{1}{4} \sqrt{\frac{\pi}{\lambda^3}}$$

$$\oint \mathbf{f} \cdot d\mathbf{l} = \iint (\nabla \times \mathbf{f}) \cdot d\mathbf{A}, \nabla \times \nabla g = 0$$

$$j_x = -\frac{i\hbar}{2m} \left[ \psi^* \frac{\partial}{\partial x} \psi - \psi \frac{\partial}{\partial x} \psi^* \right]$$

$$\langle n_\epsilon \rangle_{FD} = \frac{1}{\exp(\epsilon / k_B T) + 1} \quad \langle n_\epsilon \rangle_{BE} = \frac{1}{\exp(\epsilon / k_B T) - 1}$$

$$\langle E \rangle = -\frac{\partial \ln Z}{\partial \beta} \quad \langle M \rangle = \frac{1}{\beta} \frac{\partial \ln Z}{\partial H} \quad F = -k_B T \ln Z \quad PV = Nk_B T$$

$$\Omega(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega s} K(s) ds \quad K(s) = \int_{-\infty}^{\infty} e^{-i\omega s} K(\omega) d\omega$$

**There are two questions.**

**Show all your working and attach extra sheets if needed.**

1. (a) In one dimensional diffusion, the flux across an area is given by

$$J(x,t) = -D \frac{\partial}{\partial x} n(x,t),$$

where  $n(x,t)$  is the concentration of particles at a position  $x$  at time  $t$ . Derive the diffusion equation relating the time rate of change of  $n$  to the gradient of  $J$ . Justify your reasoning.

$J(x,t) = n(x,t) \langle v \rangle$ . The no. particles in  $[x_1, x_2]$  is  $\int_{x_1}^{x_2} dx n(x,t)$ . Conservation of mass means

$$-\frac{d}{dt} \int_{x_1}^{x_2} dx n(x,t) = -J(x_2,t) + J(x_1,t)$$

$$\approx -J(x_2,t) - \left( \frac{\partial J}{\partial x} \right) dx \Big|_{x_2} + J(x_1,t) + \left( \frac{\partial J}{\partial x} \right) dx \Big|_{x_1}$$

$$= 2 dx \frac{\partial J}{\partial x} (x_1, t)$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} n = +2D \frac{\partial^2}{\partial x^2} n}$$

(b) The probability that a molecule with velocity between  $v_1$  and  $v_1 + dv$  is scattered by a molecule with velocity  $v$  is given by

$$\left[ \tau^{-1}(v_1) \right] = \iiint [|\vec{v}_1 - \vec{v}|] [\sigma_0] [D(\vec{v}) dv_x dv_y dv_z]$$

where  $D(\vec{v})$  is the Maxwell velocity distribution. Explain the physical meaning of each bracketed term.

$\frac{1}{\tau} \equiv$  prob rate for scattering of particle with vel  $\vec{v}_1$

$|\vec{v}_1 - \vec{v}| \equiv$  Magnitude relative vel between incident & target particles

$\sigma_0 \equiv$  prob of scattering in area units

$D(\vec{v}) d^3v \equiv$  prob target with vel  $\vec{v}$  in range  $\vec{v}, \vec{v} + d^3\vec{v}$ .

(c) The solution  $n(x,t)$  (representing the number of molecules within a slice between  $x$  and  $x+dx$  at time  $t$ ) for a semi-infinite cylinder of air with cross section  $A$  and  $N$  molecules initially located at

$x=0$  is  $n(x,t) = \frac{N}{A \sqrt{\pi D t}} \exp\left(\frac{-x^2}{4Dt}\right)$ . Use this to write down an integral expression for the

average position  $\langle x \rangle$  at time  $t$ . Show, without solving the integral, that  $\langle x \rangle$  is proportional to  $t^{1/2}$ .

$$\langle x \rangle = \frac{\int_{-\infty}^{\infty} x e^{-x^2/(4Dt)} dx}{\int_{-\infty}^{\infty} e^{-x^2/(4Dt)} dx}$$

Let  $u = x (4Dt)^{-1/2}$ . Then  $du = (4Dt)^{-1/2} dx$ , and

$$\langle x \rangle = \frac{\int_{-\infty}^{\infty} u (4Dt)^{1/2} e^{-u^2} du (4Dt)^{1/2}}{\int_{-\infty}^{\infty} e^{-u^2} du (4Dt)^{1/2}}$$

OC  $t^{1/2}$

2. The Boltzmann equation in the relaxation time approximation is given by

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \dot{\mathbf{v}} \cdot \frac{\partial f}{\partial \mathbf{v}} = -\frac{f - f_0}{\tau}$$

(a) Show that this equation is satisfied by the equilibrium distribution of velocities given by

$$f_0(v_x, v_y, v_z) = n \left( \frac{m}{2\pi k_B T} \right)^{3/2} e^{-\frac{m(v_x^2 + v_y^2 + v_z^2)}{2k_B T}}$$

At equilibrium,  $\frac{\partial f_0}{\partial t} = 0$

If  $\dot{\mathbf{v}}$  independent of  $\mathbf{r}$  (as it must be with no external forces at equilibrium),  $\frac{\partial}{\partial \mathbf{r}} f = 0$ .

Also, no forces  $\Rightarrow \dot{\mathbf{v}} = 0$ .

Thus  $0 = -\frac{1}{\tau} (f_0 - f_0) = 0$

(b) Consider the Langevin equation

$$\frac{dv}{dt} = -\gamma v + \frac{1}{m} F(t)$$

where  $F(t)$  is a rapidly varying, random, force. Write  $F(t)$  and  $v$  in terms of Fourier integrals, and show how their Fourier coefficients must be related to satisfy the Langevin equation. Use the result

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \langle v(0)v(s) \rangle e^{-i\omega s} ds = \frac{k_B T}{\pi m} \frac{\gamma}{\gamma^2 + \omega^2}$$

to derive a form of the Nyquist theorem.

$$\text{Let } v(t) = \int_{-\infty}^{\infty} e^{-i\omega t} v(\omega) d\omega$$

$$F(t) = \int_{-\infty}^{\infty} e^{-i\omega t} F(\omega) d\omega$$

$$\Rightarrow -i\omega v(\omega) = -\gamma v(\omega) + \frac{F(\omega)}{m}$$

$$\Rightarrow v(\omega) = \frac{F(\omega)/m}{\gamma - i\omega}$$

$$\text{Form } \langle v^*(\omega) v(\omega) \rangle = \frac{\langle F^*(\omega) F(\omega) \rangle / m^2}{\gamma^2 + \omega^2}$$

$$\Rightarrow \frac{\langle F^*(\omega) F(\omega) \rangle}{\omega^2 (\gamma^2 + \omega^2)} = \frac{k_B T}{\pi m} \frac{\gamma}{\gamma^2 + \omega^2}$$

$$\Rightarrow \boxed{\langle F^*(\omega) F(\omega) \rangle = \gamma \frac{\omega}{\pi} k_B T}$$