NAME ANSWERS
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2008 THIRD YEAR STATISTICAL MECHANICS TEST A

THE FOLLOWING INFORMATION MAY BE USEFUL

$\hbar = 1.05\times10^{-34}$ J s $= 6.6\times10^{-16}$ eV s, 1 eV $= 1.60\times10^{-19}$ J

electron mass $m_e = 9.11\times10^{-31}$ kg , neutron mass $m_n = 1.67\times10^{-27}$ kg

$k_B = 1.381 \times 10^{-23}$ J/K, $N_A = 6.022 \times 10^{23}$ mol^{-1}

$$e^{iax} = \cos(ax) + i\sin(ax),\ \cos(ax) = \frac{1}{2}\left(e^{iax} + e^{-iax}\right),\ \sin(ax) = \frac{1}{2i}\left(e^{iax} - e^{-iax}\right)$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots \qquad N! \cong N^N e^{-N}\sqrt{2\pi N} \qquad \sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}$$

$$\int_0^\infty e^{-\lambda x^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}} \qquad \int_0^\infty x e^{-\lambda x^2}dx = \frac{1}{2\lambda} \qquad \int_0^\infty x^2 e^{-\lambda x^2}dx = \frac{1}{4}\sqrt{\frac{\pi}{\lambda^3}}$$

$$\oint \mathbf{f}\cdot d\mathbf{l} = \iint\left(\nabla\times\mathbf{f}\right)\cdot d\mathbf{A},\ \nabla\times\nabla g = 0$$

$$j_x = -\frac{i\hbar}{2m}\left[\psi^*\frac{\partial}{\partial x}\psi - \psi\frac{\partial}{\partial x}\psi^*\right]$$

$$\langle n_\varepsilon\rangle_{FD} = \frac{1}{\exp(\varepsilon / k_B T) + 1} \qquad \langle n_\varepsilon\rangle_{BE} = \frac{1}{\exp(\varepsilon / k_B T) - 1}$$

$$\langle E\rangle = -\frac{\partial \ln Z}{\partial\beta} \quad \langle M\rangle = \frac{1}{\beta}\frac{\partial \ln Z}{\partial H} \quad F = -k_B T \ln Z \quad PV = Nk_BT$$

$$\Omega(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega s}K(s)ds \qquad K(s) = \int_{-\infty}^{\infty} e^{-i\omega s}K(\omega)d\omega$$

There are two questions.
Show all your working and attach extra sheets if needed.

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1. (a) In one dimensional diffusion, the flux across an area is given by

$$J(x,t) = -D\frac{\partial}{\partial x}n(x,t),$$

where $n(x,t)$ is the concentration of particles at a position x at time t. Derive the diffusion equation relating the time rate of change of n to the gradient of J. Justify your reasoning.

$J(x,t) = n(x,t)\langle v\rangle$. The no. particles in region $[x_1, x_2]$ is $\int_{x_1}^{x_2} dx\, n(x,t)$. Conservation of mass & particles means

$$-\int_{x_1}^{x_2} dx \frac{\partial}{\partial t} n(x,t) = -J(x_1,t) + J(x_2,t)$$

$$\simeq -J(x_0,t) - \ell\frac{\partial}{\partial x}J(x,t)\Big|_{x=x_0} + J(x_0,t) + \ell\frac{\partial}{\partial x}J(x,t)\Big|_{x=x_0}$$

$$= 2\ell\frac{\partial}{\partial x}J(x_0,t)$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}n = +2\ell D\frac{\partial^2}{\partial x^2}n}$$

(b) The total scattering cross section for an electron-air molecule collision is about ~~10^{-15}~~ σ_0 cm^2. At what gas pressure will 90% of the electrons emitted from a cathode reach an anode ~~20 cm~~ a distance ℓ away? (Assume that any electron scattered out of the beam does not reach the anode.)

$$0.1\left(\frac{\text{no. inc.}}{\text{time}}\right) = \left(\frac{\text{no. inc.}}{\text{time}\cdot\text{area}}\right)\sigma_0\left(\text{no. targets}\right)$$

$$\Rightarrow 0.1 = \frac{\sigma_0}{a}N = \frac{\sigma_0}{a}\left(\frac{PV}{k_BT}\right) = \boxed{\frac{\sigma_0 P}{k_BT}\ell}$$

since $V = a\ell$;

Thus

$$\boxed{P = \frac{k_BT}{10\sigma_0\ell}}$$

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(c) The solution $n(x,t)$ (representing the number of molecules within a slice between x and $x+dx$ at time t) for a semi-infinite cylinder of air with cross section A and N molecules initially located at $x=0$ is $n(x,t) = \frac{N}{A\sqrt{\pi D t}}\exp\left(\frac{-x^2}{4Dt}\right)$. Use this to write down an integral expression for the average position $\langle x\rangle$ at time t. Show, *without solving the integral,* that $\langle x\rangle$ is proportional to $t^{1/2}$.

$$\langle x \rangle = \frac{\int_{-\infty}^{\infty} x\, e^{-x^2/(4Dt)}\, dx}{\int_{-\infty}^{\infty} e^{-x^2/(4Dt)}\, dx}$$

Let $u = \frac{x}{\sqrt{4Dt}}$. Then $du = (4Dt)^{-1/2}\, dx$; and

$$\langle x \rangle = \frac{\int u (4Dt)^{1/2} e^{-u^2}\, du\, (4Dt)^{1/2}}{\int e^{-u^2}\, du\, (4Dt)^{1/2}} \quad \boxed{\propto t^{1/2}}$$

2. The Boltzmann equation without collisions is given by

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \dot{\mathbf{v}}\cdot\frac{\partial f}{\partial \mathbf{v}} = 0.$$

(a) Use the Boltzmann equation to derive the conservation of mass law,

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho \mathbf{v}),$$

where ρ is the number of particles at position $\mathbf{r}$ at time t.

Form $\int d^3v\, m D f$ where $D = \frac{\partial}{\partial t} + \vec{v}\cdot\frac{\partial}{\partial \vec{r}} + \dot{\vec{v}}\cdot\frac{\partial}{\partial \vec{v}}$.

If no forces present, $\dot{\vec{v}} = 0$. Then

$$\int d^3v\, m \frac{\partial}{\partial t} f = \int d^3v \frac{\partial}{\partial t} m f = m\frac{\partial}{\partial t}\int d^3v\, f = m\frac{\partial}{\partial t}\rho$$

and

$$\int d^3v\, m\vec{v}\cdot\frac{\partial}{\partial \vec{r}} f = m\int d^3v \frac{\partial}{\partial \vec{r}}\cdot(\vec{v} f) - m\int d^3v\, f \underbrace{\left(\frac{\partial}{\partial \vec{r}}\cdot\vec{v}\right)}_{\to 0}$$

$$= m\frac{\partial}{\partial \vec{r}}\cdot\int d^3v\, \vec{v} f = m\frac{\partial}{\partial \vec{r}}\cdot\langle \rho\vec{v}\rangle$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial \vec{r}}\cdot\langle \rho\vec{v}\rangle}$$

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(b) Consider the Langevin equation

$$\frac{dv}{dt} = -\frac{1}{\tau}v + \frac{1}{m}F(t)$$

where $F(t)$ is a rapidly varying, random, force.

i) Derive the expression

$$v(t) = v(0)e^{-t/\tau} + e^{-t/\tau}\int_0^t e^{u/\tau}\,\frac{F(u)}{m}\,du.$$

Let $t \to u$, Multiply by $e^{u/\tau}$ and integrate:

$$\int_0^t du\, e^{u/\tau}\frac{dv}{du} = -\frac{1}{\tau}\int_0^t du\, e^{u/\tau} v + \frac{1}{m}\int_0^t du\, F(u) e^{u/\tau}$$

$$e^{t/\tau}v(t) - v(0) - \frac{1}{\tau}\int_0^t du\, e^{u/\tau} v = -\frac{1}{\tau}\int_0^t du\, e^{u/\tau} v + \frac{1}{m}\int_0^t du\, F(u) e^{u}$$

$$\Rightarrow \boxed{v(t) = e^{-t/\tau}v(0) + \frac{e^{-t/\tau}}{m}\int_0^t e^{u/\tau}F(u)\,du}$$

ii) Starting with the equation given in i), show that $\langle v(t)\rangle = v(0)e^{-t/\tau}$. Derive an expression for $\langle v(t)^2\rangle$. Explain what is meant by 'memory' in the resulting equation.

With $\langle F(u)\rangle = 0$, $\langle v(t)\rangle = e^{-t/\tau}\langle v(0)\rangle$
$= e^{-t/\tau}\, v(0)$

For $\langle v^2\rangle$, start with

$$v^2 = e^{-2t/\tau}v^2(0) + e^{-2t/\tau}v(0)\int_0^t e^{u/\tau}F(u)\,du + \frac{e^{-2t/\tau}}{m^2}\int_0^t e^{u/\tau}\int_0^t e^{u'/\tau}F(u)F(u')\,du\,du'$$

Average, use $\langle F(u)\rangle = 0$. Then

$$\langle v^2\rangle = v^2(0)e^{-2t/\tau} + \frac{1}{m^2}e^{-2t/\tau}\underbrace{\int_0^t\int_0^t du\,du'\, e^{\frac{1}{\tau}(u+u')}\langle F(u)F(u')\rangle}_{\text{memory}}$$

$\boxed{\text{Memory} \Rightarrow v^2(t) \text{ depends on } F \text{ at earlier times.}}$