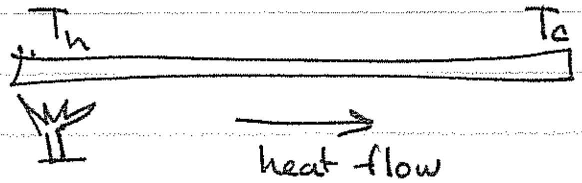
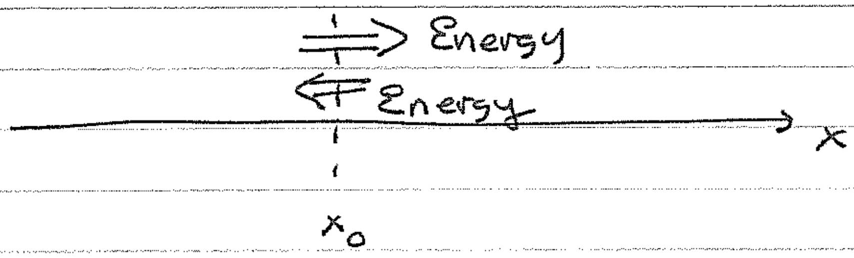
Other Diffusion Processes

(1) Thermal Conduction:

Temperature gradient drives transfer of energy



Consider flux of energy across x_0:



$$\mathcal{E}\text{-flux} = \left(\frac{\text{no. "particles"}}{\text{area} \cdot \text{time}}\right)\left(\begin{array}{c}\text{average energy}\\ \text{per particle}\end{array}\right)$$

$$= n \underbrace{v}_{\langle v \rangle} \left(\tfrac{1}{2} k_B T\right)$$

Equipartition (in 1 D)

Now $T(x)$, so compare flux at $x_0 \pm \ell$:

$$\underset{\text{at } x_0}{\text{Nett } \mathcal{E}\text{-flux}} = \frac{1}{2} n v k_B \Big(T(x_0+\ell) - T(x_0-\ell) \Big)$$

$$\simeq \frac{1}{2} n v k_B \left(+2\ell \frac{d}{dx} T\right)_{x_0}$$

$$\equiv Q \quad \leftarrow \text{heat transferred}$$

Important features:

- the form is $Q = \kappa \frac{\partial}{\partial x} T$

  where κ = thermal conductivity constant

- $\kappa \propto n \langle v \rangle \ell = \frac{\langle v \rangle}{\sqrt{2}\, \sigma_0}$

  $\Rightarrow$ depends on scattering cross section

  $\Rightarrow$ depends on $T^{1/2}$ through $\langle v \rangle$:

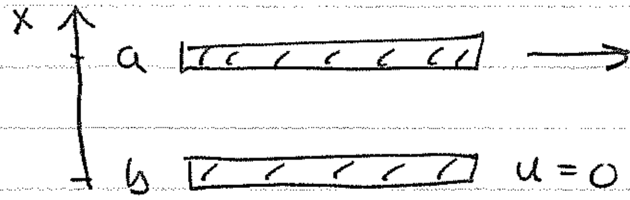
$$\langle v \rangle = \sqrt{\frac{8}{\pi} \frac{k_B T}{m}}$$

for ideal gas of density n_s.

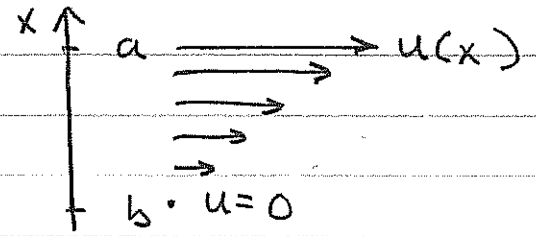
- Note: these 1 dimensional models are artificial. Careful 3 dimensional models produce different numerical prefactors (i.e, $\kappa = \frac{1}{3\sqrt{2}} c \langle v \rangle$; $c = \left(\frac{\partial E}{\partial T}\right)_V$).

2. Viscosity: diffusion of momentum

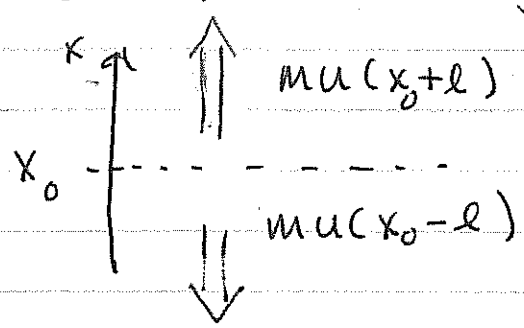
Suppose two plates – one moves



Viscous fluid => top follows plate a, bottom follows plate b. Velocity grading:



Fluid has mass, so have momentum gradient and diffusion along x:



Change in momentum => force => pressure so

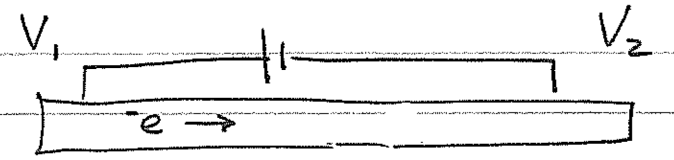
$$P(x_0) = n \overset{v}{\langle v \rangle} \left[mu(x_0+\ell) - mu(x_0-\ell) \right]$$

$$\Rightarrow \boxed{P(x) = \underbrace{\ell n m \langle v \rangle}_{\eta \,\propto\, T^{1/2}/\sigma} \frac{\partial}{\partial x} u(x)}$$

$\Rightarrow$ Experiment confirms that κ & η vary as $T^{1/2}$ — same physics for two different phenomena

3. Electrical Conduction

Current of charge caused by voltage difference:



$\left(\frac{V_1 - V_2}{x_1 - x_2} \Rightarrow \mathcal{E} \right)$ $\longrightarrow x$

Here current J along x is

$$J = n e v = n e \langle V \rangle$$

But charges move, between collisions, under $\mathcal{E}$:

$$m \frac{d}{dt} V = e\mathcal{E}$$

$$\Rightarrow \quad V = \frac{e\mathcal{E}}{m} t + v(0)$$

Now average to form $v = \langle V \rangle$:

$$\langle V \rangle = v = \frac{e\mathcal{E}}{m} \langle t \rangle + \langle V(0) \rangle$$

Expect

$$\langle V(0) \rangle = 0 \quad \text{for random collisions}$$

Know

$$\langle t \rangle = \int_0^\infty t \, \frac{1}{\tau} e^{-t/\tau} \, dt = \tau$$

So

$$v = \frac{e}{m} \mathcal{E} \tau$$

$$\Rightarrow \quad J = \frac{ne^2}{m}\tau = \sigma_{elec} \mathcal{E}$$

conductivity

Or

$$\sigma_{elec} = \frac{ne^2\tau}{m} = \frac{ne^2}{m} \frac{1}{\tilde{n} v \sigma_0}$$

no. targets ≠ scatterers

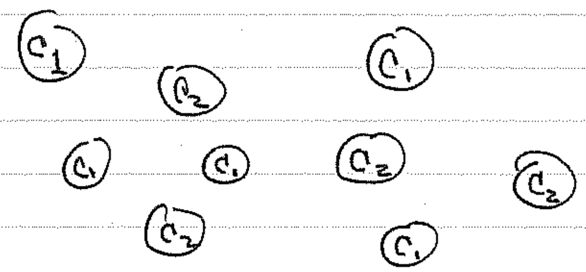
$$\Rightarrow \quad \boxed{\sigma_{elec} \propto T^{-1/2}}$$

conductivity → ∞ as T → 0

(4) Pattern formation

<u>Example</u>: skin patterns

One model is that morphogen molecules diffuse across skin with uniform distribution pigment cells (embryo skin)



two potential cell colours 1 & 2

- Suppose c_1 produce inhibitor morphogens (suppress colour 1 cells) & activators (convert colour 2 cells to colour 1)
- Let inhibitor have larger ℓ (mean path)
- Nett effect on a cell depends on sum of inhibitor & activator concentrations: $n_1 - n_2$

$$\Rightarrow \quad \frac{\partial}{\partial t} m(\vec{r}, t) = \nabla \cdot \overleftrightarrow{D} \cdot \nabla m - km + Q$$

morphogen concentration $n_1 - n_2$; anisotropic diffusion; decay; production

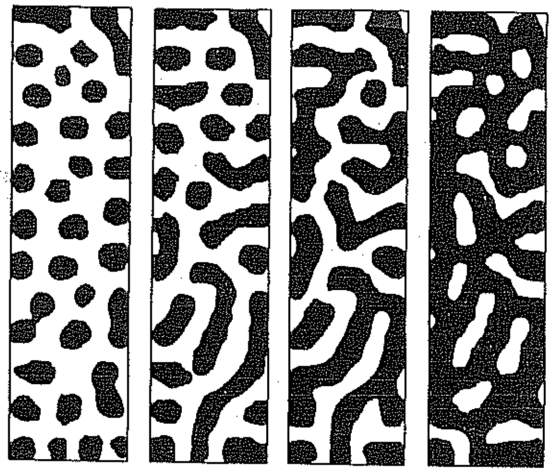


Figure 2.14 Skin Patterns Emerging from Different Inhibitor Values

Example computer simulation of resulting patterns for different thresholds for cell activation.

A General Theory: applicable to dynamic pattern formation (turbulence)

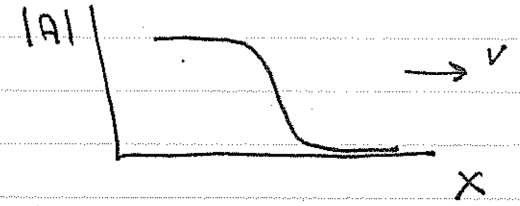
Pattern generation classified by equation type. Example: Complex Landau Ginzburg Eq

$$\tau_0 \frac{\partial}{\partial t} A = \xi_0^2 \frac{\partial^2}{\partial x^2} A + \epsilon A - g|A|^2 A$$

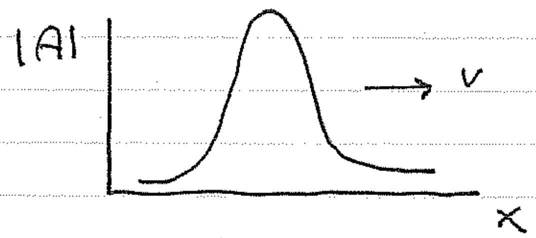
$$A \sim \underbrace{a e^{ik_c x} + a^* e^{-ik_c x}}_{\mathbb{R}\text{e fields}} + \cdots$$

solutions include

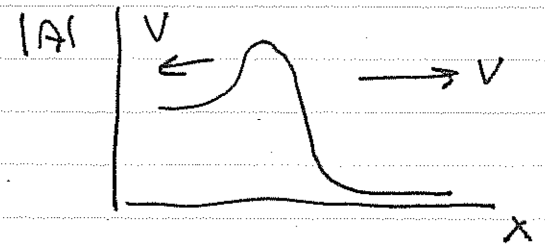
fronts



pulses



sources



sinks