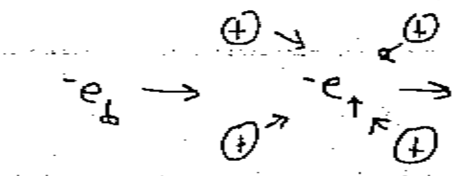
Charged Superfluids : Superconductors

Electrons are fermions and cannot condense into a single energy state if there are more than 2. However it is possible for electrons of opposite spin to interact in an elastic material via phonons in such a way as to behave like zero spin bosons.



⇐ effective attraction between $-e_\uparrow$ and $-e_\downarrow$

The complete interacting system can then condense to a ground state. Note that here, as well as for superfluids, the condensation is in <u>momentum</u> not position. This is the fundamental difference between bose condensation and classical condensation.

Landau theory can be used to describe superconductivity in direct analogy to how we described superfluidity.

We begin by redefining mass and charge to be consistent with the microscopic pairing of electrons responsible for the effective attraction (mediated by phonons):

$$2m \to m$$

$$2e \to e$$

Characteristic features of superconductors involve a magnetic field. In a normal metal, a field $\vec{B}$ can penetrate easily through a metal. This is not true in a superconductor and is the reason why magnets can be levitated, superconducting quantum interference devices work, and how magnetic vortex lattices form.

To see how this works, we first need to know how a magnetic field appears in the hamiltonian of an electron. This sounds like a puzzle because $\vec{B}$ can not do work on a free electron, but it does modify the total momentum.

The hamiltonian of free particle in electric and magnetic fields is written as

$$H = \frac{1}{2m}\left(p - p_{field}\right)^2 + e\varphi$$

where φ is the electrostatic potential, and

$$\vec{p} = \vec{p}_{kin} + \vec{p}_{field}$$

where $\vec{p}_{kin} = m\vec{v}$ and p_{field} is the momentum associated with the magnetic field:

$$\vec{P}_{field} = \frac{e}{c}\vec{A}$$

Note; in $\nabla \cdot A = 0$ gauge, from $P_{field} = \frac{1}{4\pi c}\int dV\, \vec{E} \times \vec{B}$

Here $\vec{A}$ is the vector potential: $\vec{B} = \nabla \times \vec{A}$. The particle momentum appearing in H is the conjugate to the position variable $\vec{q}$ as defined by the Lagrangian for an electron in an electromagnetic field:

$$\vec{p} = \frac{\partial}{\partial \dot{\vec{q}}}\mathcal{L}$$

The Lagrangian that can give the correct equations of motion is

$$\mathcal{L} = \frac{1}{2}m\dot{\vec{q}}^{\,2} - e\varphi + \frac{e}{c}\dot{\vec{q}} \cdot \vec{A}$$

where $\varphi(\vec{q})$ and $\vec{A}(\vec{q})$. The correct equations of motion are of the form

$$m\ddot{x} = eE_x + \frac{e}{c}(\hat{v}\times\vec{B})_x$$

$$m\ddot{y} = eE_y + \frac{e}{c}(\hat{v}\times\vec{B})_y$$

$$m\ddot{z} = eE_z + \frac{e}{c}(\hat{v}\times\vec{B})_z$$

when

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}$$

$$\vec{B} = \nabla\times\vec{A}$$

Note: these can be verified using $\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}} - \frac{\partial\mathcal{L}}{\partial q} = 0$.

The hamiltonian is then

$$H = \vec{p}\cdot\dot{\vec{q}} - \mathcal{L}$$

where

$$\vec{p} = \frac{\partial}{\partial\dot{\vec{q}}}\mathcal{L} = m\dot{\vec{q}} + \frac{e}{c}\vec{A}.$$

Because the momentum operator therefore involves $\vec{A}$ in addition to the kinetic velocity $\vec{v}$, we define a superfluid velocity as

$$\vec{V}_s = \frac{1}{m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)$$

As an operator on an order parameter Ψ, we define

$$\hat{\vec{V}}_s = \frac{1}{m}\left(-i\hbar\nabla - \frac{e}{c}\vec{A}\right)$$

So that with a complex φ,

$$\varphi = a\, e^{iS}$$

when both a and S depend on position,

$$\hat{\vec{v}}_s \varphi = \frac{\hbar}{m}\left(\nabla S - \frac{e}{\hbar c}\vec{A} \right) \varphi .$$

We therefore write the velocity of the superconductor, in analogy to the superfluid, as

$$\vec{v}_s = \frac{\hbar}{m}\left(\nabla S - \frac{e}{\hbar c}\hat{A} \right) .$$

Because $\nabla \times \nabla S$ is identically zero, we have now

$$\nabla \times \vec{v}_s = - \frac{e}{mc} \nabla \times \hat{A} = - \frac{e}{mc}\vec{B}$$

This is called the London equation. From this we can demonstrate how a superconductor "screens" magnetic fields.

First we define a current as the charge number en_s times $\vec{v}_s$:

$$\vec{j} = en_s \vec{v}_s$$

so that

$$\nabla \times \vec{j} = - \frac{n_s e^2}{mc}\vec{B}$$

We can next obtain a differential equation for $\vec{B}$ by using the Maxwell equation

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} \right)$$

Forming

$$\nabla \times \nabla \times \vec{B} = \mu_0 \nabla \times \vec{J}$$

$$= \mu_0 \left(- \frac{n_s e^2}{m c} \right) \vec{B}$$

Since

$$\nabla \times \nabla \times \vec{B} = \nabla (\nabla \cdot \vec{B}) - \nabla^2 \vec{B},$$

with $\nabla \cdot \vec{B} = 0$,

$$\nabla^2 \vec{B} = \frac{1}{\lambda_L^2} \vec{B}$$

where

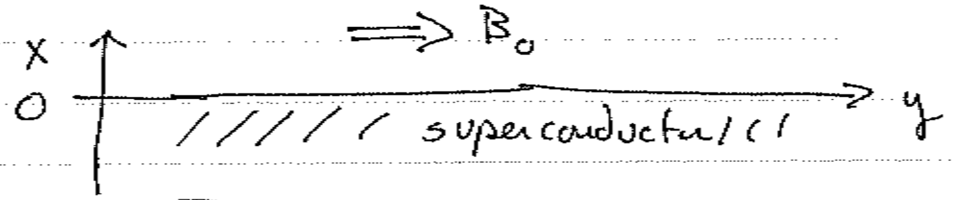
$$\lambda_L = \left(\frac{\epsilon_0 m c^2}{n_s e^2} \right)^{1/2}$$

is a length that depends on the number of charge carriers, n_s.

The first consequence of this equation is that $\vec{B}$ = constant is not possible. That is, $\vec{B} = \vec{B}_0$ as a uniform field is not a solution inside the superconductor unless $\vec{B}_0 = 0$.

The inability to have a uniform magnetic field inside a superconductor is called the Meissner effect.

A magnetic field can penetrate some distance into a superconductor however. Consider a field B_0 applied in the y direction parallel to the surface of a superconductor.



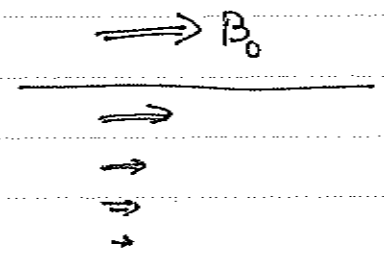
Put the surface in the y-z plane at $x=0$. The equation $\nabla^2 \vec{B} = \frac{1}{\lambda_L^2} \vec{B}$ becomes

$$\frac{\partial^2}{\partial x^2} B = \frac{1}{\lambda_L} B \qquad \text{for} \qquad x < 0$$

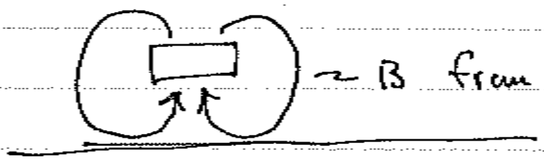
which has solution

$$B(x) = B(0)\, e^{\pm x/\lambda_L}$$

We choose the $+x$ solution for $x<0$, and see that B decays into the superconductor a characteristic distance λ_L, eventually vanishing to $\emptyset$ as $x \to \infty$.

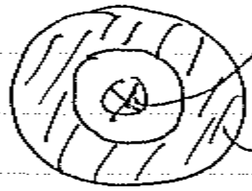


The field is "screened" by supercurrents and does not penetrate much further than λ_L from the surface. This is why magnets can "float" above the surface of a superconductor:



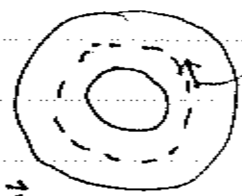
Fluxons

Lastly we mention a useful phenomena associated with superconducting rings. Suppose a ring with no current is subjected to a magnetic field:



If the current is zero in the ring, then a path integral (well away from the ring surface) is zero:

$$0 = \oint \vec{v}_s \cdot d\vec{l}$$



$$= \oint \frac{\hbar}{m}\left(\nabla S - \frac{e}{\hbar c}\vec{A}\right)\cdot d\vec{l}$$

$\Rightarrow$

$$\oint \nabla S \cdot d\vec{l} = \frac{e}{\hbar c} \oint \vec{A} \cdot d\vec{l}$$

But by Stoke's theorem,

$$\oint \vec{A} \cdot d\vec{l} = \int_{\text{surface}} \vec{B} \cdot d\vec{a} = \Phi$$

where Φ is the magnetic flux through the centre of the ring.

According to our argument for superfluids, we can change S by $2\pi n$ upon a closed path around the loop. Thus

$$\Phi = \frac{\hbar c}{e} 2\pi n$$

where n is an integer. The allowed flux through the loop is quantized and the unit of these "fluxons" is

$$\Phi_0 = \frac{2\pi \hbar c}{e} \cong 2.0678 \times 10^{-15} \text{ tesla} \cdot \text{m}^2$$

The Superconducting QUantized Interference Device (SQUID) uses this feature to measure very small magnetic fields.

Also, in analogy to vortex lines in superfluids, magnetic field lines can penetrate a superconductor in quantized units of Φ_0. These can even form lattices of line arrays.

Superconductor Phenomena

- Can destroy with large magnetic field $H \geq H_c$

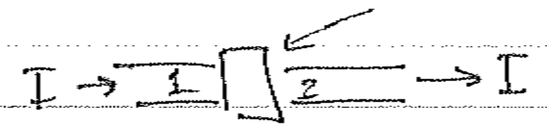
- Cooper pairs: if state $\vec{k}_\uparrow$ occupied, so is state $-\vec{k}_\downarrow$. States always occupied in pairs.

- Energy gap: 1^{st} excited state has energy
$$\underset{\text{ground state}}{\mathcal{E}_1 = \mathcal{E}_0} + \Delta \leftarrow \text{gap}$$

- Type I: $\lambda_L \sim 0$ for all $H < H_c$ (perfect Meissner Effect)

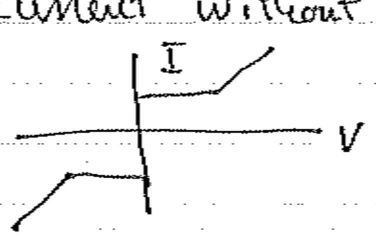
- Type II: $\lambda_L \sim 10^{-6}$ cm (Pd, Nb, Al)
Different from Type I in terms of surface energy at normal / superconducting phases (1 is positive, the other negative).

- Josephson effects: tunnel barrier

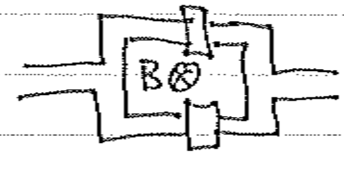


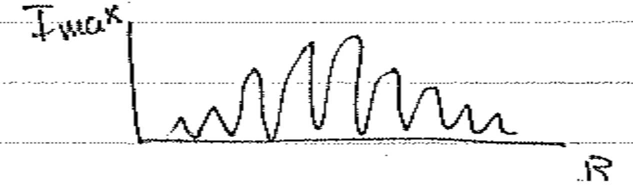
(occurs via $\Delta\varphi = \varphi_1 - \varphi_2$)

- dc allows current without voltage!



- ac gives peak oscillation (use for SQUID)





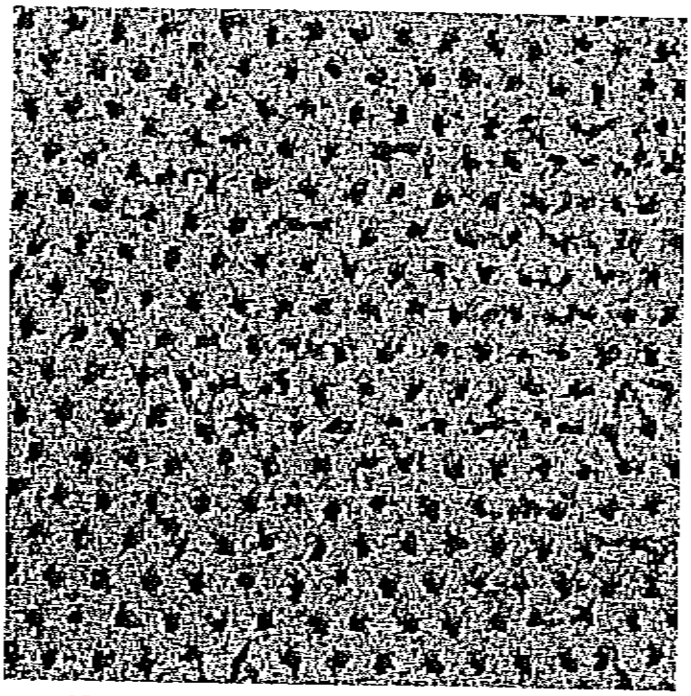
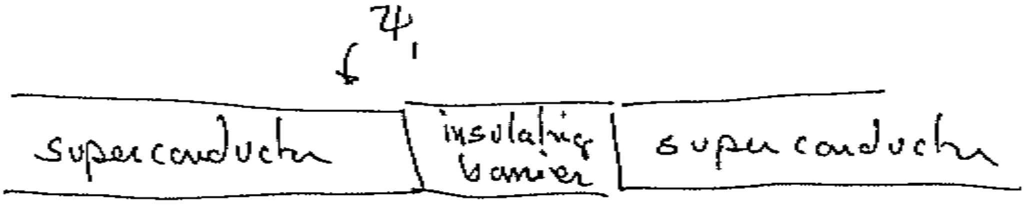


Figure 19 Triangular lattice of fluxoids through top surface of a superconducting cylinder. The points of exit of the flux lines are decorated with fine ferromagnetic particles. The electron microscope image is at a magnification of 8300, by U. Essmann and H. Träuble.

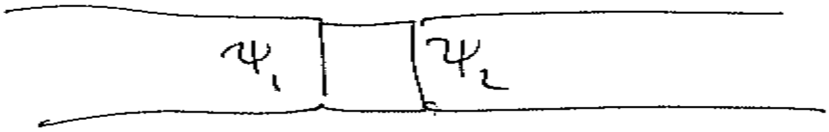
DC Josephson Effect

Let ψ_1 be the probability amplitude to find a superconducting electron pair at one side of an interface:



No current can flow "over" the barrier unless an applied voltage is greater than the energy of a pair, $E_g/2e$.

If the barrier is thin, tunneling can occur. Let ψ_2 be the probability amplitude on the other side of the barrier:



If the tunnel rate is R, then the Schrödinger equation is

$$i\hbar \frac{\partial}{\partial t}\psi_1 = \hbar R \psi_2$$

$$i\hbar \frac{\partial}{\partial t}\psi_2 = \hbar R \psi_1$$

Let the Ψ have an amplitude related to number density n, and some phase φ:

$$\Psi_1 = \sqrt{n_1}\, e^{i\varphi_1}$$
$$\Psi_2 = \sqrt{n_2}\, e^{i\varphi_2}$$

Then
$$\frac{\partial}{\partial t}\Psi_1 = \frac{1}{2} n_1^{-1/2} e^{i\varphi_1}\, \dot{n}_1 + i n_1^{1/2} e^{i\varphi_1}\, \dot{\varphi}_1 = \frac{R}{i}\Psi_2$$
$$\frac{\partial}{\partial t}\Psi_2 = \frac{1}{2} n_2^{-1/2} e^{i\varphi_2}\, \dot{n}_2 + i n_2^{1/2} e^{i\varphi_2}\, \dot{\varphi}_2 = \frac{R}{i}\Psi_1$$

or

$$\frac{1}{2} n_1^{-1/2}\, \dot{n}_1 + i n_1^{1/2}\, \dot{\varphi}_1 = -iR\, n_2^{1/2}\, e^{i(\varphi_2 - \varphi_1)}$$
$$\frac{1}{2} n_2^{-1/2}\, \dot{n}_2 + i n_2^{1/2}\, \dot{\varphi}_2 = -iR\, n_1^{1/2}\, e^{i(\varphi_1 - \varphi_2)}$$

or
$$\frac{1}{2}\dot{n}_1 + i n_1 \dot{\varphi}_1 = -iR\, \tilde{n}\, e^{i\delta}$$
$$\frac{1}{2}\dot{n}_2 + i n_2 \dot{\varphi}_2 = -iR\, \tilde{n}\, e^{-i\delta}$$

where
$$\tilde{n} = \sqrt{n_1 n_2}$$
$$\delta = \varphi_2 - \varphi_1$$

Equating real and imaginary parts,

$$\dot{n}_1 = 2R\tilde{n}\sin\delta$$
$$\dot{n}_2 = -2R\tilde{n}\sin\delta$$

and

$$n_1\dot{\varphi}_1 = -R\tilde{n}\cos\delta$$
$$n_2\dot{\varphi}_2 = -R\tilde{n}\cos\delta$$

If there is no applied voltage and the superconductors are identical, $n_1 \simeq n_2$, so

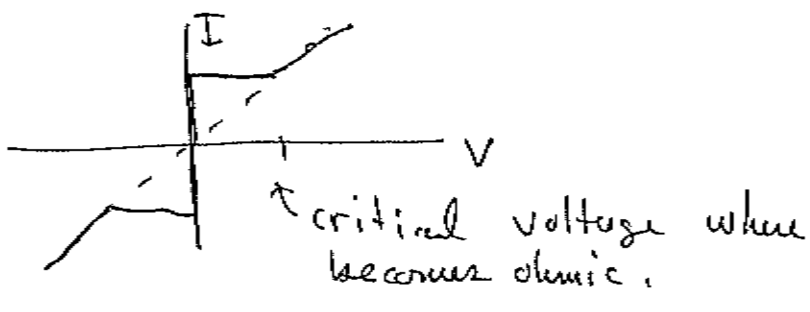
$$\dot{\varphi}_1 = \dot{\varphi}_2 \Rightarrow \frac{\partial}{\partial t}(\varphi_2 - \varphi_1) = 0$$

and
$$\dot{n}_1 = -\dot{n}_2$$

Thus phase relative is constant, but a decrease in n_1 is exactly balanced by an increase in n_2, and this happens spontaneously. Hence, a current flows:

$$J \propto \dot{n}_1 \Rightarrow \boxed{J = J_0 \sin\delta}$$

The I-V curve looks like



Exercise: Consider a constant voltage applied across the junction, with amplitude less than the critical value (for ohmic conduction). This adds a term $-eV\Psi_1$ to the $\frac{\partial}{\partial t}\Psi_1$ equation, and a term $+eV\Psi_2$ to the $\frac{\partial}{\partial t}\Psi_2$ equation:

$$i\hbar \frac{\partial}{\partial t}\Psi_1 = \hbar R \Psi_2 - eV\Psi_1$$

$$i\hbar \frac{\partial}{\partial t}\Psi_2 = \hbar R \Psi_1 + eV\Psi_2$$

Show that the resulting current is given by

$$J = J_0 \sin\left[\delta(t=0) - 2eVt/\hbar\right]$$

where t is time and $\delta = \varphi_2 - \varphi_1$. This means that a d.c. voltage produces an a.c. current with frequency $\frac{2eV}{\hbar}$.