Landau-Ginzburg Theory & Superfluidity

The free energy written as a power law expansion in m for the Ising model is a special case of a more general Landau-Ginzburg theory. In this theory, group theoretic arguments are used to establish the validity of a power law expansion of the free energy near a critical point.

The expansion is made in terms of an averaged quantity $\varphi(r)$ that provides a measure of collective "order" in the system of particles. For spin $\frac{1}{2}$ particles, this can be $m = \langle \sigma_z \rangle$. For gases it may be density.

The expansion is made for the free energy density, and the total free energy is an integral over the leading terms in the expansion. To order four, this is

$$F = \int d^3r \left\{ F_0 + \frac{1}{2} a |\varphi|^2 + \frac{1}{4} b |\varphi|^4 + \frac{1}{2} \lambda |\nabla \varphi|^2 \right\}$$

The free energy depends on parameters b, λ and F_0 that ultimately depend on details of the particular physical system. The second order coefficient a depends on temperature as before,

$$a = \alpha (T - T_c)$$

The parameters α and T_c are also specific to particular physical systems.

When $T < T_c$, the free energy is minimized by a uniform $\varphi(r)$ such that $\varphi(r) = $ constant and $\nabla\varphi = 0$. The minima occur when

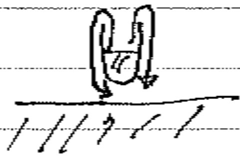
$$\frac{\partial}{\partial\varphi}\left(F_0 + \frac{1}{2}a|\varphi|^2 + \frac{1}{4}b|\varphi|^4\right) = 0$$

$$\Rightarrow \quad |\varphi|^2 = -\frac{a}{b}$$

Excitations above this lowest energy state involve spatial variations in $\varphi(r)$, and thereby involve a new energy from the λ term.

Ginzburg-Pitaevskii theory of superfluidity

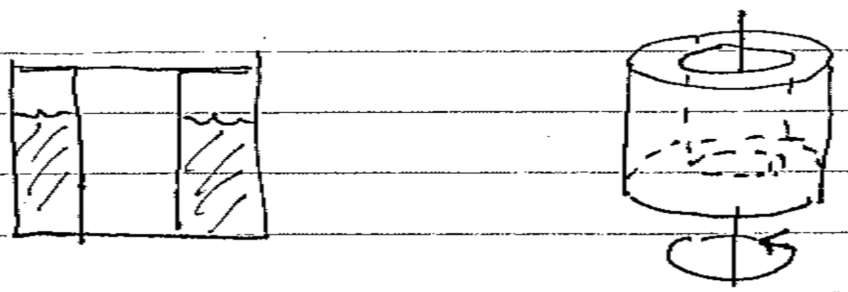
A physical interpretation of this λ term is central to the Ginzburg-Pitaevskii theory of superfluidity. Superfluid ^4He appears at temperatures below 2.18 K and displays some remarkable properties:

- frictionless flow through capillaries
-  "super leak"
- spontaneous "creep" 

· reduced moment of inertia 

We begin a description of the theory by considering an example for the reduced rotational inertia.

Consider a hollow ring cylinder with ^4He in the hollow:



Below 4.2 K, the ^4He is liquid. The cylinder is rotated, and a measurement can be made to determine the moment of inertia I (by measuring the period of oscillations, for example).

A normal fluid will eventually move with the same angular velocity as the container due to friction with the walls and internal viscosity. Indeed, between 2.18 K and 4.2 K, the liquid ^4He does this and one obtains an inertia I_c.

Below 2.18 K, the measured I is less than I_c! The reason is that a fraction of the fluid no longer moves with the container,

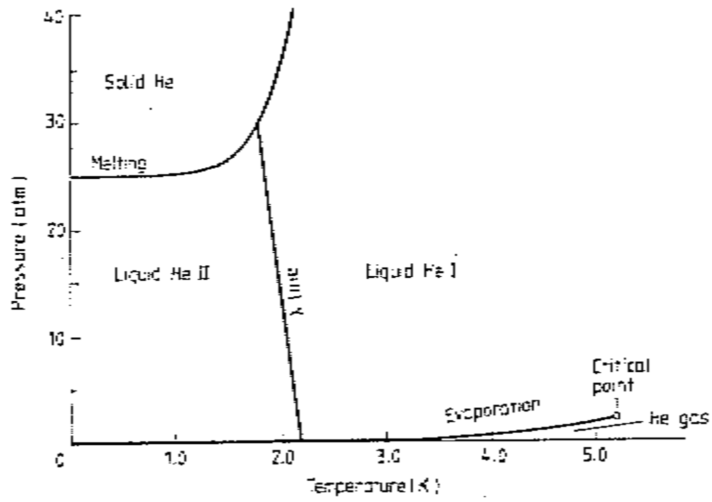


Figure 1.1 Phase diagram of ^{4}He (after London 1954).

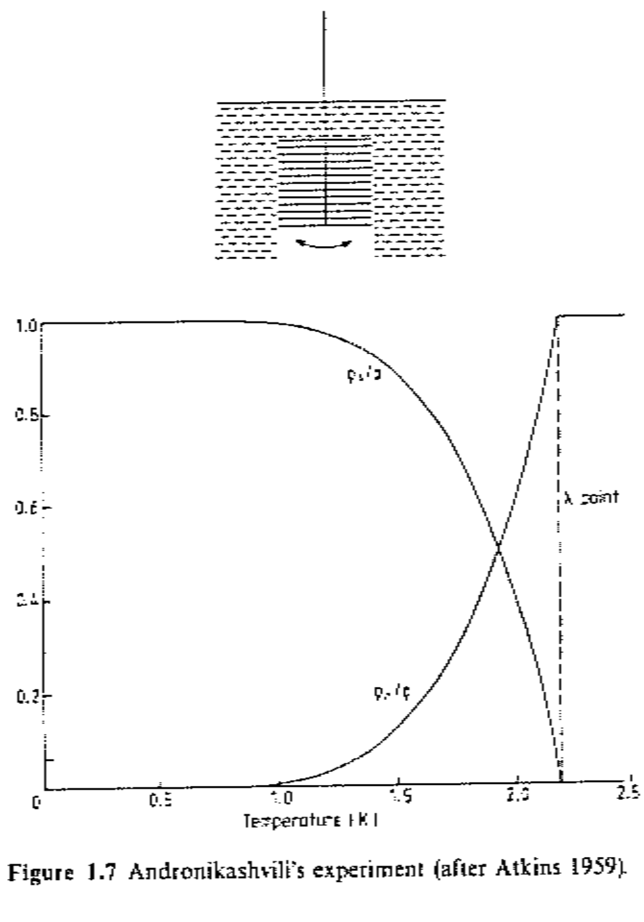


Figure 1.7 Andronikashvili's experiment (after Atkins 1959).

and therefore does not contribute to the rotational inertia. The stationary fraction is called the "superfluid" because it has zero viscosity.

The conventional description is a two-fluid model where the superfluid density ρ_s and normal fluid density ρ_n satisfy

$$\frac{\rho_n}{\rho_s + \rho_n} = \frac{I}{I_c}$$

Above 2.18 K $\quad \rho_s \cong 0$.

In Ginzburg theory, the order parameter φ is identified with ρ_s as a number density:

$$\rho_s(r) = |\varphi(r)|^2$$

This is the same as what we do in quantum mechanics with a complex wavefunction. Here φ is also allowed to be complex, and is defined as

$$\varphi(r) \equiv A(r)\, e^{iS(r)}$$

Both $A(r)$ and $S(r)$ are real functions.

The free energy density is written in the <u>rest frame</u> of the cylinder as

$$f = \frac{a}{2} A(r)^2 + \frac{b}{4} A(r)^4 + \lambda(\nabla A(r))^2 + \lambda A(r)^2 (\nabla S(r))^2$$

The parameter λ is defined as $\lambda \equiv \frac{\hbar^2}{2m}$.

If the cylinder is rotating, and the superfluid is not contributing to the rotational inertia, then it must have a velocity in the rest frame of the cylinder since it is stationary in the laboratory frame. The probability current allows us to determine the velocity.

We first write

$$j = -\frac{i\hbar}{2m}\left(\varphi^* \nabla \varphi - \varphi \nabla \varphi^*\right)$$

$$= -\frac{i\hbar}{2m}\left[A^2(i\nabla S) - A^2(-i\nabla S)\right]$$

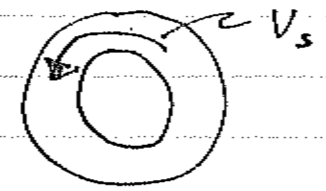
$$= \frac{\hbar}{m} A^2 \nabla S$$

The velocity of the fluid is then (note v_s & j are vectors)

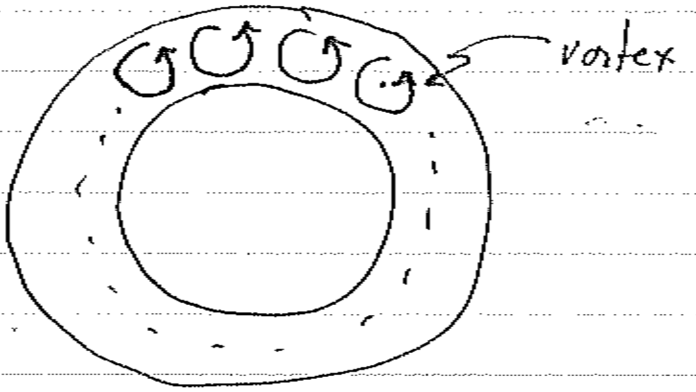
$$v_s = \frac{j}{\rho_s} = \frac{\frac{\hbar}{m} A^2 \nabla S}{A^2} = \frac{\hbar}{m}\nabla S$$

Now at this point we might be tempted to say that the cylinder rotates while the superfluid remains stationary because of a relative velocity V_s due to a spatial change in phase S (that is, $\nabla S \neq 0$).

This turns out to be partly correct, but must be corrected for how the superfluid actually rotates. The rotation is not simply uniform from the cylinder's frame:



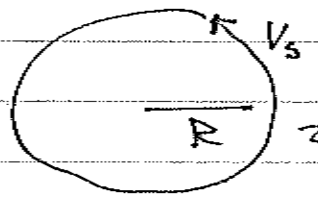
Instead, the superfluid breaks into "vortex excitations" spaced a few millimeter apart:



The center of these vortices are lines of normal fluid.

The fascinating aspect of this is that the vortices are quantized in units of h/m.

To see this, consider a closed path for flow of the superfluid:



$\approx$ path = circle radius R

Now we sum V_s over this path:

$$\oint V_s \cdot dr = \frac{\hbar}{m} \oint \nabla S \cdot dr$$

We require that $\Phi(r) = \Phi(r + 2\pi R)$ so that S can change by integer multiple of 2π without affecting Φ. Then

$$\oint \nabla S \cdot dr = \oint dS = 2\pi n \quad , n = 0, \pm 1, \pm 2 \ldots$$

so that

$$\oint V_s \cdot dr = \frac{h}{m} n \quad ; \; n = 0, \pm 1, \pm 2, \pm 3 \ldots$$

This is similar to Bohr's atomic model condition on angular momentum:

$$\oint p \, dq = n h \, .$$

The circulation is quantized, but there is an important point. Because v_s depends on the gradient of a scalar S, we can see from Stoke's theorem that

$$\oint v_s \cdot dr = \frac{\hbar}{m} \oint \nabla S \cdot dr$$

$$= \frac{\hbar}{m} \int_{\text{surface}} \underbrace{(\nabla \times \nabla S)}_{\emptyset \text{ by identity}} \cdot da$$

$$= 0$$

It therefore seems that $n = 0$ is the only possibility! The fluid is irrotational which provides an argument for zero viscosity.
(Since there can not be a 'diffusion' of velocity as in viscosity $\Rightarrow$)

The answer is that the above argument applies only for regions enclosed by the surface integral where v_s is defined everywhere (a simply connected topology). A vortex line of normal fluid defines a region where v_s is not defined, so then one can have $n \neq 0$ circulations. These circulations are vortex "excitations" in that there is an associated kinetic energy given by

$$f_{K.E.} = \lambda A^2 (\nabla S)^2 = \frac{\hbar^2}{2m} \rho_s \left(\frac{m}{\hbar} v_s\right)^2$$

$$= \left(\frac{1}{2} m v_s^2\right) \rho_s \, .$$